Approximate asymptotic integration of a higher order water-wave equation using the inverse scattering transform

A. R. Osborne

Nonlinear Processes in Geophysics (1997) 4: 29–53

Nonlinear Processes in Geophysics

Approximate asymptotic integration of a higher order water-wave equation using the inverse scattering transform

A.R. Osborne

Dipartimento di Fisica Generale dell'Università, Via Pietro Giuria 1, Torino 10125, Italy

Received 5 December 1996 - Accepted 8 March 1997

Abstract. The complete mathematical and physical characterization of nonlinear water wave dynamics has been an important goal since the fundamental partial differential equations were discovered by Euler over 200 years ago. Here I study a subset of the full solutions by considering the irrotational, unidirectional multiscale expansion of these equations in shallow-water. I seek to integrate the first higher-order wave equation, beyond the order of the Korteweg-deVries equation, using the inverse scattering transform. While I am unable to integrate this equation directly, I am instead able to integrate an analogous equation in a closely related hierarchy. This new integrable wave equation is tested for physical validity by comparing its linear dispersion relation and solitary wave solution with those of the full water wave equations and with laboratory data. The comparison is remarkably close and thus supports the physical applicability of the new equation. These results are surprising because the inverse scattering transform, long thought to be useful for solving only very special, low-order nonlinear wave equations, can now be thought of as a useful tool for approximately integrating a wide variety of physical systems to higher order. I give a simple scenario for adapting these results to the nonlinear Fourier analysis of experimentally measured wave trains.

1 Introduction

The inverse scattering transform (IST) is a modern method of mathematical physics which has been quite successful for studying nonlinear wave motions in a wide variety of physical systems. The method was discovered by Kruskal and co-workers about 30 years ago (Gardner et al., 1967) and has since been applied to a large number of physically interesting cases, including the Korteweg-deVries (KdV), the nonlinear Schroedinger (NLS) and the Kadomtsev-Petviashvili (KP) equations (Ablowitz and Segur, 1981; Calogero and Degasperis, 1982; Dodd, et. al, 1982; Novikov, et al, 1984; Newell, 1985; Drazin and Johnson, 1989). These nonlinear, Hamiltonian wave equations are all derivable from water wave dynamics in an asymptotic sense and are thus "generic" or "special" and are known to be "completely integrable" by IST. In spite of many remarkable successes the inverse scattering transform has *not* been useful for probing the general structure of the fundamental nonlinear partial differential equations (PDEs) of physics. These PDEs include the Euler equations for surface and internal wave motions in a fluid, the Navier-Stokes equations for general driven/damped fluid motions, the Maxwell equations for electromagnetic waves in dispersive media, the Einstein equations for the description of the gravitational field and for the propagation of gravitational waves, etc. (Fordy, 1990). In all of these cases, together with many others not discussed here, IST enables one to integrate the dynamical motion *only* to leading order. The integration by IST of physically important nonlinear wave equations at higher order has thus been an illusive goal.

The focus of the present paper is to provide a new approach which, for the particular case of irrotational, unidirectional motion in shallow water, allows one to extend integrability to higher order using IST. The method in this context is not the precise "asymptotically integrable" situation which occurs at leading order, but instead is an approximate integration of the second order equation obtained by means of a multiscale expansion of the fundamental nonlinear PDEs for water wave dynamics. As shown herein, the results are quite satisfactory from the point of view of the exploration of many aspects of the phase space of solutions of the nonlinear water wave equations.

The Korteweg-deVries equation (Korteweg and deVries, 1895), which describes small-but-finite-amplitude waves in shallow water, is known to be completely integrable by the inverse scattering transform (Gardner et al., 1967). The higher order model of Kaup (1975) is also integrable, although KdV is probably preferable as a model equation for a number of reasons (Kaup, 1986). More recently an extended KdV equation has also been found to be integrable (the Camassa-Holm (CH) equation) (Camassa and Holm, 1993; Camassa et al., 1994). Thus the question arises: Can other higher order nonlinear shallow water wave equations also be integrable using IST? It has been suggested that this is so (Kodama, 1985a,b; Fokas and Liu, 1996), although up to the present time I know of no other successful efforts to carry out the integration of other particular higher order, physically interesting wave equations in the shallow water regime. In order to further explore this possibility it is worthwhile discussing the main PDEs which enter into the present discussion of shallow water wave dynamics.

It is well known that an expansion of the water wave equations near zero wave number ($k \sim 0$, long waves in shallow water) gives the familiar KdV equation:

$$\eta_t + c_o \eta_x + \alpha \eta \eta_x + \beta \eta_{xxx} = 0 \tag{1}$$

Here $c_o = \sqrt{gh}$, $\alpha = 3c_o / 2h$, $\beta = c_o h^2 / 6$, where h is the water depth and g is the acceleration of gravity. A simple transformation (e.g. $u' = \lambda \eta$, $x' = x - c_o t$, $t' = \beta t$, $\lambda = \alpha / 6\beta$) reduces KdV to standard form

$$u_t + 6uu_x + u_{xxx} = 0 \tag{2}$$

where the primes have been dropped. The KdV equation is referred to here as W1, i.e. the first nonlinear equation which is obtained in the Whitham multiscale expansion of the water wave equations (Whitham, 1974) (see Section 4 for a discussion). The inverse scattering technique for solving the KdV equation may be viewed as a generalization of linear Fourier analysis for solving nonlinear wave equations. This means that the mathematical and physical structure of KdV may be computed from the appropriate Lax pair (Lax, 1968; Ablowitz and Segur, 1981; Calogero and Degasperis, 1982; Dodd, et al., 1982; Novikov et al., 1984; Newell, 1985; Drazin and Johnson, 1989; see also Section 6).

In the present paper the multiscale expansion of the water wave equations depends upon two scales of the motion, $\alpha = a / h$, $\beta = (h / l)^2$, where for concreteness I take $O(\alpha) \sim O(\beta) \sim O(\varepsilon)$ (the Kruskal principle of maximal balance). Here a and l are the characteristic wave amplitude and wave length, respectively. The multiscale expansion of the water wave equations to one order higher than the KdV equation gives (Whitham, 1974):

$$\begin{aligned} &u_t + 6uu_x + u_{xxx} + \\ &+\varepsilon(\alpha_1 u_{5x} + \alpha_2 uu_{xxx} + \alpha_3 u_x u_{xx} + \alpha_4 u^2 u_x) + O(\varepsilon^2) = 0 \end{aligned} \tag{3a}$$

Herein this equation is referred to as W2; physically it consists of the KdV equation plus $O(\varepsilon)$ corrections. In the normalization used here (see Section 4) the parameters α_i are simple constants: $\alpha_1 = 19/10$, $\alpha_2 = 10$, $\alpha_3 = 23$ and $\alpha_4 = -6$.

It is convenient to factor out α_1 in (3a) and to set $\varepsilon' = \alpha_1 \varepsilon$, $\alpha_1' = 1$, $\alpha_2' = \alpha_2 / \alpha_1$, $\alpha_3' = \alpha_3 / \alpha_1$ and $\alpha_4' = \alpha_4 / \alpha_1$. Then W2 becomes (after dropping the primes):

$$\begin{aligned} &u_t + 6uu_x + u_{xxx} + \\ &+\varepsilon(u_{5x} + \alpha_2 uu_{xxx} + \alpha_3 u_x u_{xx} + \alpha_4 u^2 u_x) + O(\varepsilon^2) = 0 \end{aligned} \tag{3b}$$

The rescaled constant coefficients are: $\alpha_1 = 1$, $\alpha_2 = 100/19$, $\alpha_3 = 230/19$ and $\alpha_4 = -60/19$. It is this particular normalization of W2 which will be studied herein.

It is worth remarking at this juncture that (3b), for the *particular set of coefficients*, $\alpha_1 = 1$, $\alpha_2 = 10$, $\alpha_3 = 20$ and $\alpha_4 = 30$, is the integrable equation found by Kodama (1985a,b):

$$\begin{aligned} &u_t + 6uu_x + u_{xxx} + \\ &+\varepsilon(u_{5x} + 10uu_{xxx} + 20u_x u_{xx} + 30u^2 u_x) + O(\varepsilon^2) = 0 \end{aligned} \tag{4}$$

Ideally one would like to extend integrability to include the physical wave equation (3b) which describes the dynamics of both surface (Korteweg and deVries, 1895) and internal waves (Lee and Beardsley, 1974), and therefore requires different values for the coefficients α_i in each of these two cases, i.e. the α_i depend upon "environmental parameters" such as the water depth, gravitational acceleration, density stratification, etc. The main focus of this paper is to show how one can approximately integrate (3b) using the inverse scattering transform.

For the work given herein it is important to formally state the definition of the "equivalence" of two different partial differential equations. As a simple example consider the KdV equation (1) and the related equation due to Benjamin, Bona and Mahoney (BBM) (1972):

$$\eta_t + c_o \eta_x + \alpha \eta \eta_x - (\beta / c_o) \eta_{xxt} = 0 \tag{5}$$

This latter equation is found by using the leading order linear behavior, $\eta_x \cong -(1/c_o)\eta_t$, in the second order dispersive term (η_{xxx}) of the KdV equation. Thus KdV (1) and BBM (5) are equivalent because one can be obtained from the other by a simple, leading order transformation. These equations are physically and mathematically the same order of approximation, i.e. they describe surface water waves to within the same accuracy.

An equation like BBM is said to be "regularized" because the presence of the time derivative in the dispersive term in (5) improves the physical behavior of the *linear dispersion relation* with respect to that for the KdV equation (Benjamin et al., 1972). An alternative perspective is that one might view BBM itself as the generic equation and hence one can carry out a "deregularization" process to derive KdV by using $\eta_t \cong -c_o \eta_x$ in the dispersive term.

The historical view is that soliton theories (the inverse scattering transform) work mainly at the leading order of approximation and are not generally applicable to arbitrary physical situations with the exception of the well-known asymptotically integrable cases such as the KdV and nonlinear Schroedinger equations. Recent work by Camassa and Holm (1993) has indicated that integrable physical structure also exists at higher order. By applying Hamiltonian techniques to the Green-Nagdy equations they found the following integrable soliton system:

$$u_t + 6uu_x + u_{xxx} - \varepsilon(u_{xxt} + 2uu_{xxx} + 4u_x u_{xx}) = 0 \qquad (6)$$

An interesting feature of this equation is that there is a time derivative in the $O(\varepsilon)$ term and thus the equation is regularized. To compare the CH equation (6) with W2 (3b) (or with the Kodama equation (4)) one can replace the $O(\varepsilon)$ time derivative by space derivatives using the leading order relation from the KdV equation: $u_t \cong -6uu_x - u_{xxx}$. One finds

$$u_t + 6uu_x + u_{xxx} + \varepsilon[u_{5x} + 4uu_{xxx} + 14u_x u_{xx}] = 0 \qquad (7)$$

Comparing this deregularized equation with (3b) reveals that the $u^2 u_x$ term is missing in (7) and the coefficients differ somewhat from those in both W2 (3b) and Kodama (4) (see Table 1). Therefore, the CH equation may be viewed in some sense as being a regularized equation lying between KdV and W2 in the infinite Whitham hierarchy (see Section 10 for further discussion and Fig. 15). Nevertheless, the CH equation, in spite of the fact that it is not one of the traditional equations from the unidirectional, multiscale expansion of the water wave equations, has physical properties which are not inconsistent with the physics of higher order waves, i. e. CH has a well-behaved linear dispersion relation ((66) below) and peaked solutions for its limiting behavior as the waves propagate into shallow water (Camassa and Holm, 1993; Camassa et al., 1994)).

The main purpose of the present work is to present a new procedure for approximately integrating nonlinear wave equations to higher nonlinear order. While the obvious *target equations* (those equations which one seeks to integrate) are those in the Whitham hierarchy, Wn (n=1,2...), I am unable to integrate these equations directly. I instead derive another class of wave equations which are formally equivalent to an *extended, regularized* Whitham hierarchy, exRWn (see Section 6

	u_{5x}	uu_{xxx}	$u_x u_{xx}$	$u^2 u_x$
Kodama	1	10	20	30
CH	1	4	14	0
W2	1	~5.3	~12.1	~-3.2

Table 1. Comparison of the constant coefficients for each $O(\varepsilon)$ term in the Kodama (4), the deregularized CH (7) and the W2 (3b) equations.

for details). By developing a *universal wave equation* associated with a *universal Lax pair* I succeed in integrating the equation exRW2 ((59) below) using inverse scattering techniques. It is found that exRW2 has certain higher order space derivatives which are replaced by time derivatives (due to the regularization process implicit in the integration procedure given in Section 6, here symbolically labelled "R"). Furthermore, additional "extended" terms beyond the $O(\varepsilon)$ approximation (symbolically labelled "ex") insure the integrability of exRW2.

As with all nonlinear wave equations derived using the method of multiple scales or other approaches, there is no *a priori* assurance that the discovery of any new equation (by whatever mathematical approach) provides an improved description of the dynamical behavior. Without appropriate testing (by numerical, analytical or experimental means) one cannot be assured that any higher order equation is better than, say, the KdV equation. Physical "closeness" of exRW2 to the water wave equations is supported by comparison of their linear dispersion relations and solitary wave solutions: these tests provide evidence for the physical correctness and robustness of the new integrable equation exRW2 (see Sections 7 and 8 below). A more severe test would be to compare the (IST) spectral structure of exRW2 to particular analytical or numerical solutions of the water wave equations and to experimental data (see below and Osborne et al, 1997).

I further discuss how the work presented here can be viewed in the light of *nonlinear Fourier analysis* (Section 9). In this context unidirectional, shallow water wave trains are representable in terms of a *basis function* which is a natural generalization of cnoidal waves to higher order; these are the travelling wave solutions of exRW2. At leading linear order (W0) the travelling wave is a sine wave; at KdV order (W1) the travelling wave is a cnoidal wave. At the order of the CH equation the travelling wave takes the form of the "peakon" as the depth $h \to 0$. *Any spectral solution of the shallow water problem, to arbitrarily high order, can be represented in terms of a linear superposition of the relevant travelling waves plus nonlinear interactions among them.* Typically the higher order travelling wave varies over a range of behaviors as it propagates adiabatically from deep to shallow water, i.e. from 1) a sine wave, 2) to a Stokes wave, 3) to a (KdV) solitary wave, 4) to a wave form which is narrower and higher than the sech^2 shape of the solitary wave and 5) finally up to near the highest (peaked) wave found by Stokes over a century ago. As is well known the CH equation describes many features of nonlinear water waves. However, while the results for higher waves are good, they are not quantitatively precise (see Section 8). Higher-order wave equations discussed herein (such as W2 and exRW2) have behavior which is *quantitatively* closer to certain properties of the water wave equations (Sections 3, 9 and 10).

An outline of this paper is now given. In Section 2 I briefly discuss nonlinear Fourier data analysis techniques at the order of the KdV equation (Osborne, 1995). These techniques are the state of the art for the nonlinear Fourier analysis of unidirectional, shallow water wave trains. In order to provide physical motivation for extending the procedure to higher order, I discuss results from a laboratory experiment in Section 3. One finds experimentally that water waves, while propagating into shallow regions, become quite higher, narrower and consequently more nonlinear than the cnoidal wave solutions of the KdV equation. These results suggest the need for higher order (hopefully integrable) theories for describing water wave dynamics. In Section 4 I review the multiscale expansion of the unidirectional, water wave equations due to Whitham and in Section 5 I discuss the recent results of Fokas and Liu (1996) for establishing the integrability of certain higher order wave equations. In Section 6 I give a systematic approach for integrating higher order equations which are in some sense close to the Whitham hierarchy. In particular I integrate the extended, regularized equation I refer to as exRW2. The linear dispersion relation of exRW2 is compared to the linear dispersion relations of the KdV, CH and water wave equations in Section 7. In Section 8 I compare the solitary wave solutions of these equations. The results suggest that exRW2 is an improved integrable equation for describing the evolution of unidirectional, shallow water waves, i.e. it has a number of physical characteristics which are superior to those of the KdV and CH equations. Section 9 discusses the consequences of adapting nonlinear Fourier analysis methods to the higher order of the exRW2 equation. The results lay the groundwork for the *natural extension to higher order* of nonlinear Fourier analysis procedures which have previously used only cnoidal wave basis functions (Osborne, 1991, 1993, 1995; Osborne and Petti, 1994; Osborne, 1995). Finally Section 10 provides a summary, a discussion of the results and a plan for future research.

2 Linear and Nonlinear Fourier Analysis

The main purpose of this Section is to provide physical perspective for the nonlinear Fourier structure for the unidirectional propagation of nonlinear waves in shallow water. It is this perspective which leads to the design of the experiments given in Section 3 and to the subsequent developments for the approximate integration of unidirectional wave dynamics to higher order given in Section 6. In this Section I the discuss nonlinear Fourier analysis of surface wave dynamics at the order of the KdV equation.

Because of the nonlinear behavior of water waves in general, it has been argued that *nonlinear* Fourier structure should be pursued and understood (Osborne, 1991; Osborne, 1995). Before addressing issues of nonlinearity, however, it is worthwhile recalling the Fourier analysis of linear waves. Fourier analysis allows the construction of linear wave trains, $\eta(x,t)$, by a linear superposition of sine waves:

$$\eta(x,t) = \sum_{n=1}^{N} \eta_n \sin(k_n x - \omega_n t + \phi_n) \tag{8}$$

In the present case there are N sine waves which are interpreted as "degrees of freedom" or "Fourier components" in the wave train. In (8) the η_n are the Fourier amplitudes, the k_n are the wave numbers, the ω_n are the frequencies and the ϕ_n are the phases. The relationship between the frequencies, ω_n, and the wave numbers, k_n, is given by the dispersion relation, written symbolically: $\omega_n = \omega_n(k_n)$. For example, the dispersion relation for *long waves in shallow water*

$$\omega = c_o k - \beta k^3 \tag{9}$$

has the associated partial differential equation (the linearized Korteweg-deVries equation or W0) given by:

$$\eta_t + c_o \eta_x + \beta \eta_{xxx} = 0 \tag{10}$$

The coefficients c_o, β are the constants given above with regard to (1). The simplest periodic solution to (10) is a travelling sine wave

$$\eta(x,t) = \eta_o \sin(k_o x - \omega_o t + \phi_o) \tag{11}$$

from which the general Fourier solution for an N component wave train may be constructed by linear superposition (8). The important point is that the *amplitudes of the sine waves and their phases are constants of the motion, provided that the wave dynamics are linear.* In oceanic applications one is often interested in the analysis of time series, i.e. measurements of the wave amplitude, $\eta(0,t)$, taken at a fixed spatial location over some convenient time interval; this implies setting $x = 0$ in (8) and (11).

In spite of the elegant simplicity of the linear theory, ocean waves are known to be nonlinear. The simplest prototypical shallow water wave equation for which nonlinearity occurs is the Korteweg-deVries equation (1). This equation is the same as (10) except for the presence of the nonlinear convective derivative term, $\eta\eta_x$, prefixed by the constant coefficient, $\alpha = 3c_o/2h$. While the general solution of (10) for periodic boundary conditions is easily found using the linear Fourier transform (8), the general IST solution to the nonlinear equation (1), for periodic boundary conditions, required an additional 170 years of mathematical progress (Whitham, 1974; Miles, 1980; Dubrovin, et al., 1976; Its and Matveev, 1975; Flaschka and MacLaughlin, 1976; Date and Tanaka, 1976).

The simple periodic travelling wave solution of KdV (1) is the *cnoidal wave*, well known in shallow water oceanography and offshore engineering (Munk, 1949; Weigel, 1964; Whitham, 1974; Miles, 1980; Sarpkaya and Isaacson, 1981):

$$\eta(x,t)=\frac{4k^2}{\lambda}\sum_{n=1}^{\infty}\frac{n(-1)^n q^n}{1-q^{2n}}\cos[nk_o(x-U_ot)+\phi_o]= \qquad (12a)$$

$$=2\eta_o cn^2\{(K(m)/\pi)[k_ox-\omega_ot+\phi_o];m\}$$

where $\lambda=\alpha/6\beta=3/2h^3$ and $q=e^{-\pi K'/K}$ is the nome. The associated dispersion relation (with nonlinear correction) is given by:

$$\omega_o=U_ok_o=c_ok_o\{1+2\eta_o/h-2k_o^2h^2K^2(m)/3\pi^2\} \qquad (12b)$$

The modulus has the following explicit form:

$$mK^2(m)=\frac{3\pi^2\eta_o}{2k_o^2h^3}=4\pi^2Ur; \qquad Ur=\frac{3\eta_o}{8k_o^2h^3} \qquad (12c)$$

η_o is the amplitude of the cnoidal wave, Ur is the Ursell number, k_o is the wave number, $c_o=\sqrt{gh}$, $K(m)$ is the elliptic integral such that $K(m)=K'(1-m)$ (Abramowitz and Stegun, 1964). Note that the series in (12a), truncated to N terms, is the shallow-water, Nth order Stokes wave. In the limit as the modulus $m\to 0$ the cnoidal wave reduces to a sine wave; when $m\to 1$ the cnoidal wave approaches a solitary wave or soliton; intermediate values of the modulus correspond to the Stokes wave (see Fig. 1).

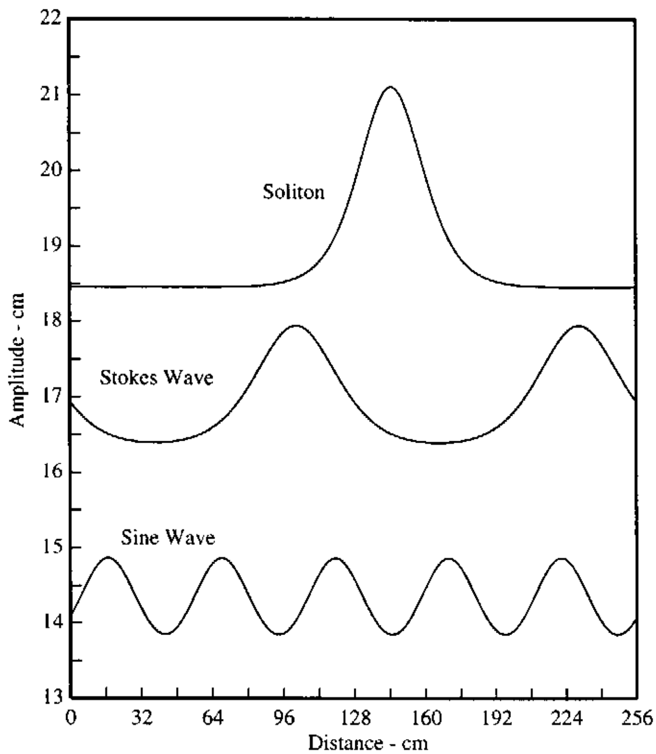

Fig. 1. Examples of cnoidal waves. In vertical order (top to bottom) are shown a solitary wave or soliton, a Stokes wave and a sine wave. Note that each example has its own unique amplitude (η_n), wave number (k_n, which controls the number of oscillations in a single period) and modulus (m, which controls the shape of the wave).

With regard to (1) (the KdV equation) and (12) (the periodic travelling cnoidal wave solution to KdV) an important problem in mathematical physics and related practical applications remained open for a century. While it is known that linear Fourier analysis (8) works well for linear wave equations (such as (10)) with sine wave basis functions (11), the more difficult question as to whether there exists a generalization of linear Fourier analysis (for nonlinear equations like (1)) with cnoidal wave basis functions (12a) remained unresolved from a theoretical point of view until about 20 years ago (Dubrovin, et al., 1976; Its and Matveev, 1975; Flaschka and MacLaughlin, 1976) and from a concrete point of view until recently (Osborne, 1995). In this latter work, nonlinear Fourier analysis has been formulated in a physical and mathematical form simple enough that practical oceanographic and engineering applications of the method can now be made. This approach is based upon the general periodic solution to the KdV equation (1) in terms of the so-called *θ-function representation*:

$$\eta(x,t)=\frac{2}{\lambda}\frac{\partial^2}{\partial x^2}\ln\Theta_N(x,t) \qquad (13a)$$

for $\lambda=\alpha/6\beta$ and

$$\Theta_N(x,t)= \qquad (13b)$$

$$=\sum_{M_1=-\infty}^{\infty}\cdots\sum_{M_N=-\infty}^{\infty}\exp\left[i\sum_{n=1}^{N}M_nX_n+\frac{1}{2}\sum_{m,n=1}^{N}M_mM_nB_{mn}\right]$$

Here N is the number of cnoidal waves in a (broad-spectrum) solution to the KdV equation. The summation indices M_n $(1\le n\le N)$ are integers summed from $-\infty$ to ∞. The θ-function phases have the same form as in linear Fourier analysis: $X_n=k_nx-\omega_nt+\phi_n$. Explicit computation of the period matrix, $\mathbf{B}=\{B_{mn}\}$, the wave numbers, k_n, the frequencies, ω_n, and the phases, ϕ_n, is discussed elsewhere (Osborne, 1995). The period matrix $\mathbf{B}$ is constant and negative definite and defines the cnoidal wave amplitudes and moduli (diagonal terms) and their nonlinear pair-wise interactions (off-diagonal terms). Equations (13) are discussed in detail for the particular case $N=2$ by Boyd (1984). On the basis of the θ-function formulation (13) one can prove the following theorem (Osborne, 1995):

Theorem of Nonlinear Fourier Analysis: The θ-function solution (13) to the KdV equation (1) can be written in the following form:

$$\eta(x,t) = \frac{2}{\lambda}\frac{\partial^2}{\partial x^2}\ln\Theta_N(x,t) = \tag{14}$$

$$= \underbrace{\eta_{cn}(x,t)}_{\substack{\text{Linear superposition}\\ \text{of cnoidal waves}}} + \underbrace{\eta_{int}(x,t)}_{\substack{\text{Nonlinear interactions}\\ \text{among the cnoidal waves}}}$$

The theorem essentially states that *Shallow water wave trains (governed by the KdV equation) can be represented by a linear superposition of cnoidal waves plus their mutual nonlinear interactions.* How is this formulation related to linear Fourier analysis? This is seen by letting the wave amplitudes become so small that the cnoidal wave components become sine waves and the nonlinear interactions tend to zero. In this way linear Fourier analysis is recovered from the nonlinear theory.

An important aspect of the above theorem is that the *amplitudes of the cnoidal waves and their phases are constants of the motion, provided that the motion is governed by the KdV equation.* It is important to recognize that the nonlinear interaction term in (14) is generally *not* a perturbation. The interactions can, for sufficiently nonlinear waves, make an $O(1)$ contribution to the dynamics. One of the aims of the present paper is to provide the theoretical underpinning for extending the above theorem to higher order so that it reads: *Shallow water wave trains can be represented by a linear superposition of higher order travelling waves plus their mutual nonlinear interactions.* These issues are discussed further in Sections 6 and 9.

I now address the synthesis of shallow-water wave trains from their constituent cnoidal waves at the order of the KdV equation. I give an example which illustrates the generality of the approach. The N-cnoidal-wave solution to KdV is given by (14), which can be written:

$$\eta(x,t) = \tag{15}$$

$$= 2\sum_{n=1}^{N}\eta_n cn^2\{(K(m_n)/\pi)[k_n x - \omega_n t + \phi_n]; m_n\} + \eta_{int}(x,t)$$

The explicit form for the interactions is (written in the obvious vector notation (Osborne, 1995)):

$$\eta_{int}(x,t) = \frac{2}{\lambda}\frac{\partial^2}{\partial x^2}\ln\left\{1 + \frac{F(\mathbf{X},C)}{F(\mathbf{X},1)}\right\}$$

where

$$F(\mathbf{X},C) = \sum_{\mathbf{M}} Ce^{i\mathbf{M}\cdot\mathbf{X}+\frac{1}{2}\mathbf{M}^T\cdot\mathbf{D}\cdot\mathbf{M}}$$

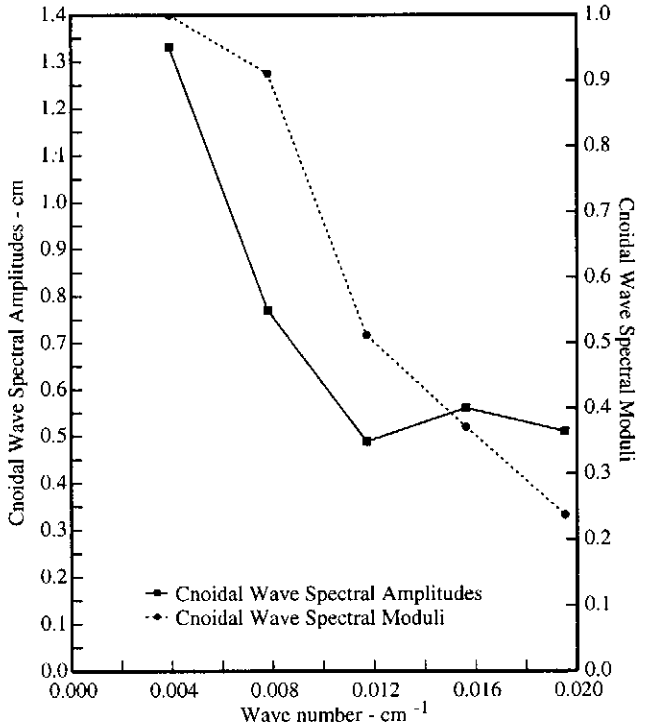

Fig. 2. An example of nonlinear Fourier analysis using the periodic inverse scattering transform. There are five cnoidal wave components in the spectrum. Shown are the amplitudes, η_n, and the moduli, m_n, of the cnoidal waves graphed as a function of wave number, k_n.

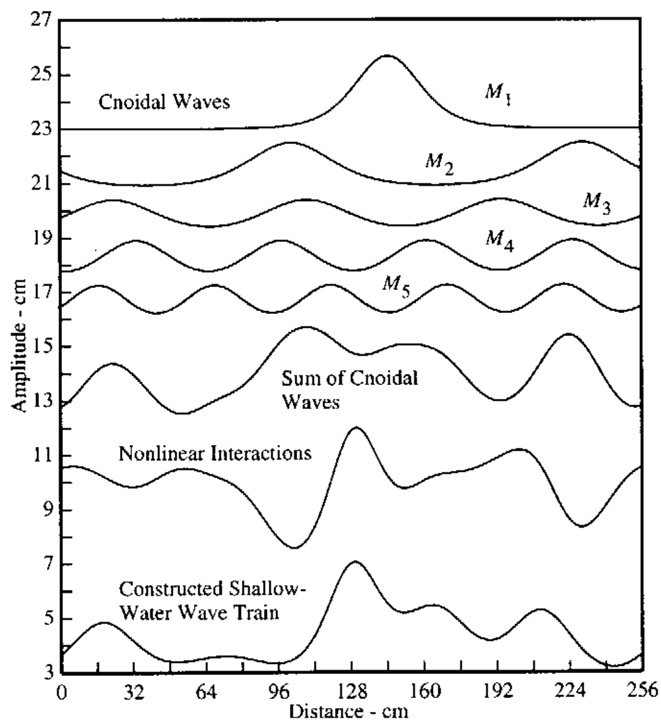

Fig. 3. The cnoidal wave components in the spectrum of Fig. 2 (vertically ordered from small to large wave numbers) are shown, together with the sum of the cnoidal waves, the nonlinear interactions and the synthesized five-component wave train. The linear superposition of the cnoidal waves plus interactions yields the constructed wave train at the bottom of the panel.

$$C = e^{\frac{1}{2}\mathbf{M}^T \cdot \mathbf{O} \cdot \mathbf{M}} - 1$$

The interaction matrix, $\mathbf{B}=\mathbf{D}+\mathbf{O}$, has been separated into diagonal, $\mathbf{D}$, and off-diagonal parts, $\mathbf{O}$. The vectors, $\mathbf{X}=\{X_n\}$ and $\mathbf{M}=\{M_n\}$, and matrices, $\mathbf{B}=\{B_{mn}\}$, $\mathbf{O}=\{O_{mn}\}$ and $\mathbf{D}=\{D_{nn}\}$, have the definitions discussed above.

Figs. 2 and 3 give an example of the synthesis of a wave train with $N=5$ cnoidal waves. The cnoidal wave spectrum is shown in Fig. 2; both the cnoidal wave amplitudes, η_n, and the moduli, m_n, are graphed as a function of wave number, k_n, $1 \le n \le 5$. The first cnoidal wave (leftmost in Fig. 2), η_1, has a modulus near 1, $m_1 \sim 1$, and hence can be interpreted as a solitary wave (see uppermost curve in Fig. 3). The next cnoidal wave has modulus 0.91 and is a Stokes wave. The remaining three waves are small amplitude Stokes waves with moduli 0.51, 0.37 and 0.23 respectively. Note that the wave numbers of the various cnoidal waves have the values $k_n = n\Delta k = 2\pi n / L$ for $L = 256$ cm and $n = 1, 2, \ldots, 5$ (in complete correspondence with the linear Fourier transform). The five cnoidal waves are shown at the top of Fig. 3 in vertical order from the lowest wave number (M_1) to the highest (M_5). Below these are the "sum of the cnoidal waves," the "nonlinear interaction contribution" and the "constructed shallow-water wave train solution" of the KdV equation. The latter KdV wave train consists of the summed cnoidal waves plus the nonlinear interactions. This example constitutes the direct application of the Cnoidal Wave Theorem given above in (14). In principle one can apply the theorem to an arbitrary number N of cnoidal waves.

It is worth emphasizing that the nonlinear interactions in Fig. 3 are not small with respect to the summed cnoidal waves. Thus the interactions cannot in general be thought of as perturbations to the nonlinear dynamics in the present example. It is well known that the nonlinear interactions of the KdV equation arise as a consequence of the *phase shifts* occurring among the nonlinear components in the spectrum. Therefore, the large amplitude oscillations in the interaction contribution are generally seen to apply a spatio-temporal phase shift to the summed component cnoidal waves.

3 Experimental Search for Higher Order Nonlinear Effects in the Laboratory

As waves propagate into shallow water regions their periods remain constant while their wave lengths shorten. As a result their crests close ranks so that a shoaling wave is much steeper than its offshore counterpart. Propagation into shallow water results in the crests becoming increasingly higher and narrower, while the troughs become shallower. All of these nonlinear effects have been known for over a century, but quantitatively assessing how important they are to all orders in the spectral analysis of nonlinear wave trains is still an open question.

This Section is devoted to preliminary results from a laboratory experiment which was designed to help understand a few requirements that one might expect of a higher order wave theory. An extensive analysis is given elsewhere (Osborne et al., 1997). The work has been conducted at the wave tank facility at the Hydraulics Section of the Department of Civil Engineering at the University of Florence (Osborne and Petti, 1994; Osborne et al., 1997). The wave tank is 46 m long with a cross section of 0.76 m by 0.80 cm and has a ramp running almost the full length of the channel with a slope of 1/100 (Fig. 4). The water depth at the wave maker is $h = 40$ cm.

For the experiments discussed here the programmable wave maker generated simple sine waves with a height of 5 cm and a period of 3 s. From Fig. 5 we see the time series of measurements taken at the 10 shallowest of the 15 probes. The first 5 probes are not shown for brevity; they are beneath Fig. 5. One sees that the first signal at the bottom of the figure has already deformed into the shape of a Stokes wave; as the wave train propagates up the ramp (moving upward in Fig. 5) the waves distort slowly, becoming narrower and higher as they evolve into shallow water. The signals near the bottom of the ramp are in relatively deep water and are small in height (5 cm). Those at the top of the figure are larger in height (8.6 cm) and occur in very shallow water. The depths at each of the probes are indicated in the figure. The probe time series shown in the figure extend from a depth of 29.6 cm (bottom), to a depth of 11.5 cm (top). The probes are roughly evenly spaced along the ramp with the exception of the last (at the top) which has been placed at a point near where the waves reached their maximum amplitude, just prior to overturning and breaking. Considerable effort was extended to place the last probe at the point of the "highest wave." A video

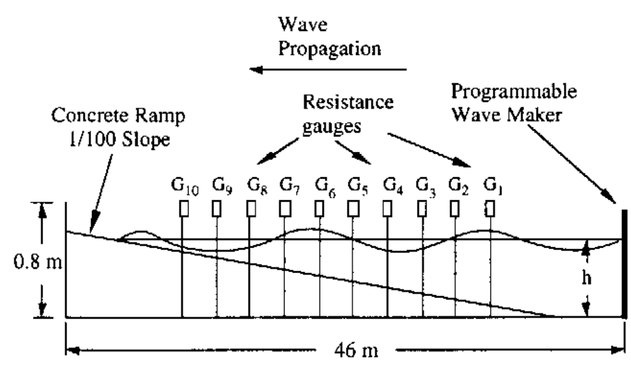

Fig. 4. Schematic of the wave tank at the Hydraulic Section of the Department of Civil Engineering, University of Florence. The facility is 46 m long and 0.76 m by 0.80 m in cross section. The test section for the experiments lies above a concrete ramp with slope of 1/100. Fifteen resistance gauges (only ten are shown) measure the wave amplitudes along the ramp as the waves propagate from 40 cm water depth (at the wave maker) to 11.5 cm depth (a few moments before the waves begin to overturn and to break).

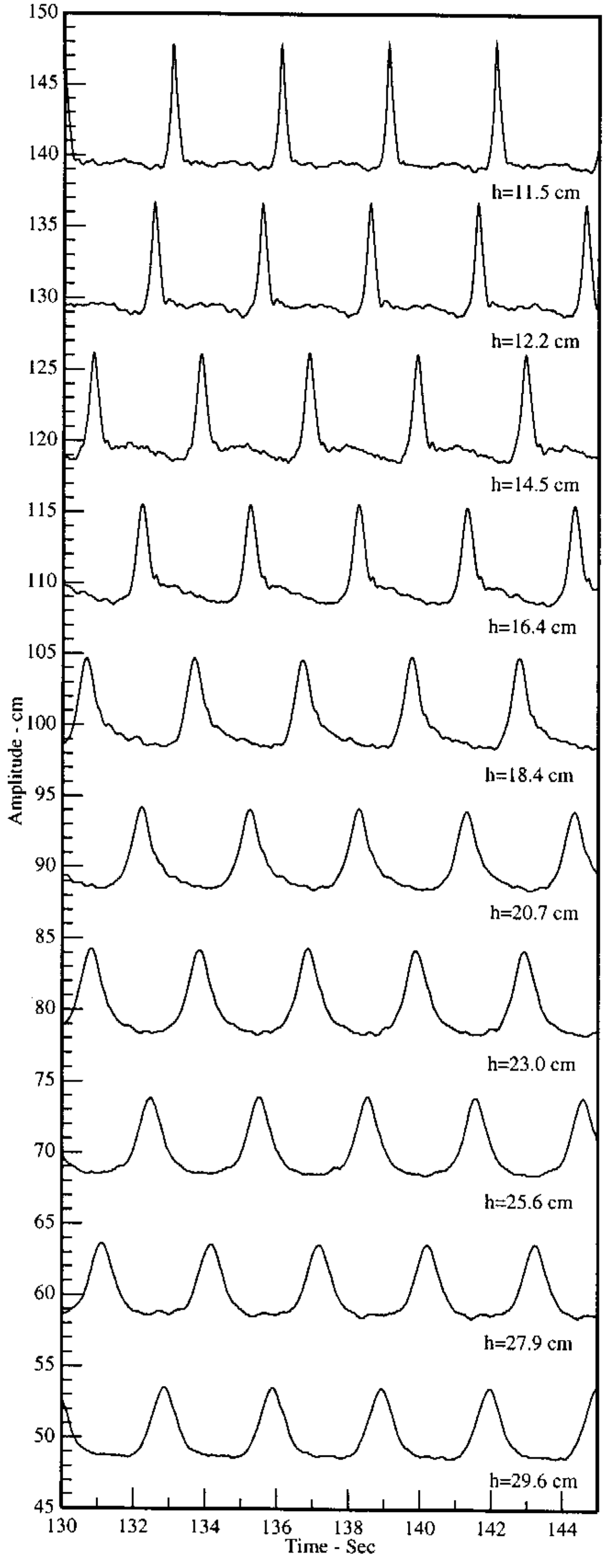

Fig. 5. Wave tank experiment in which a sine wave input (beneath the figure in a water depth of 40 cm and not shown here) propagates up the ramp with slope 1/100 and slowly deforms into highly peaked wave forms (top) which occur just at the instant before breaking. The water depths at each of the 10 probes are indicated in the figure.

camera was used to determine the instance of incipient wave breaking. This provided an estimate of the desired probe position used for the measurement of the upper signal in Fig. 5.

The wave heights in Fig. 5 are seen to nearly double during propagation from deep to shallow water, while the widths of the shallow pulses are more than a factor of two narrower than those near the wave maker. Some right/left asymmetry is visible in the recorded wave trains of Fig. 5; this effect is due primarily to the influence of the 1% bottom slope. Theoretically, this kind of near-adiabatic wave propagation should yield, in sufficiently shallow water, the Stokes highest wave, well known to have a 120^{0} angle at the peak (a result which is computed for wave propagation governed by pure potential flow on a flat bottom) (Whitham, 1974) (see discussion below with regard to Fig. 8).

It is worthwhile estimating how the Ursell number (Whitham, 1974; Miles, 1980) (an important measure of nonlinearity in shallow water waves) changes during the propagation from deep (40 cm) to shallow water (11.5 cm). The Ursell number is conveniently written in the following form:

$$Ur = \frac{3}{16\pi^2}\left(\frac{a}{h}\right)\left(\frac{c_o T}{h}\right)^2$$

Here a is the wave amplitude and T is the period. It is clear that the Ursell number increases by a factor of $(8.7\,\text{cm}\,/\,5\,\text{cm})\times(40\,\text{cm}\,/\,11.5\,\text{cm})^3 \cong 73$ as the waves propagate from the first probe to the last. The period remains constant while the amplitude increases and the depth decreases; these changes result in nearly two orders of magnitude increase in the Ursell number during the nonlinear evolution observed in the present experiment.

Let us now focus on the uppermost signal in Fig. 5. As mentioned above the probe position for this signal (water depth $h = 11.5$ cm) was moved to a particular point along the ramp very nearly where the maximum wave amplitude occurred, immediately before wave breaking began. It is estimated that each highly peaked wave (with amplitude 8.7 cm and amplitude-to-depth ratio 0.76) was measured only a few hundredths of a second before the wave form began to overturn and to break. It is in this sense that these shallowest of signals were the most nonlinear of those measured in the present experiment. I do not include here estimates of the influence of the return flow on the measured signals; this issue is addressed in detail elsewhere (Osborne et al., 1997).

An estimate of the phase speed of the larger waves can be obtained from the measurements at the last two probes. The probe separation is $L_{15-14} = 66$ cm and the time interval between peaks is 0.52 s. Thus the phase speed of a single peak has the average value 126.9

cm/s. The average depth for the two shallowest probes was 11.9 cm for which the average linear phase speed ($\sqrt{gh}$) was computed to be 108.0 cm/s.

In order to assess some of the properties of these "almost highest waves" I make the following decomposition of the measured wave train:

$$\eta(t) = \eta_{sym} + \eta_{asym}$$

where

$$\eta_{sym} = \frac{1}{2}[\eta(t-t_p) + \eta(-(t-t_p))]$$

$$\eta_{asym} = \frac{1}{2}[\eta(t-t_p) - \eta(-(t-t_p))]$$

where t_p is the time corresponding to the peak of a particular travelling wave. Thus the waves have been separated into a *symmetric part*, η_{sym}, and an *antisymmetric part*, η_{asym}. The summing and differencing in the above equations is made with respect to the *peak* of a particular wave. The symmetric part has been obtained by *averaging* the original signal with the time-reversed signal (relative to a single peak). The antisymmetric part results from the *difference* of these two signals.

The above perspective has been motivated by Svendsend and Hansen (1978) where it is shown to leading order that solitary waves on a sloping bottom have a symmetric part plus an antisymmetric part. The symmetric part is an estimate of the "true solitary wave" propagating on a flat bottom; the antisymmetric part occurs due to the presence of the bottom slope. Using the expression $\eta_{sym} = [\eta(t-t_p) + \eta(-(t-t_p))]/2$ one obtains a smoothed, symmetric estimate of a particular peaked wave form, an example of which is shown in Fig. 6, for a two second interval centered on a single solitary wave. This figure emphasizes one of the most important properties of the peaked waves measured at the last two probes, i.e. that the waves rise very quickly out of the background "radiation" to form a sharp peak and then rapidly fall once again into the background.

Is this peaked-wave form close to the simple solitary

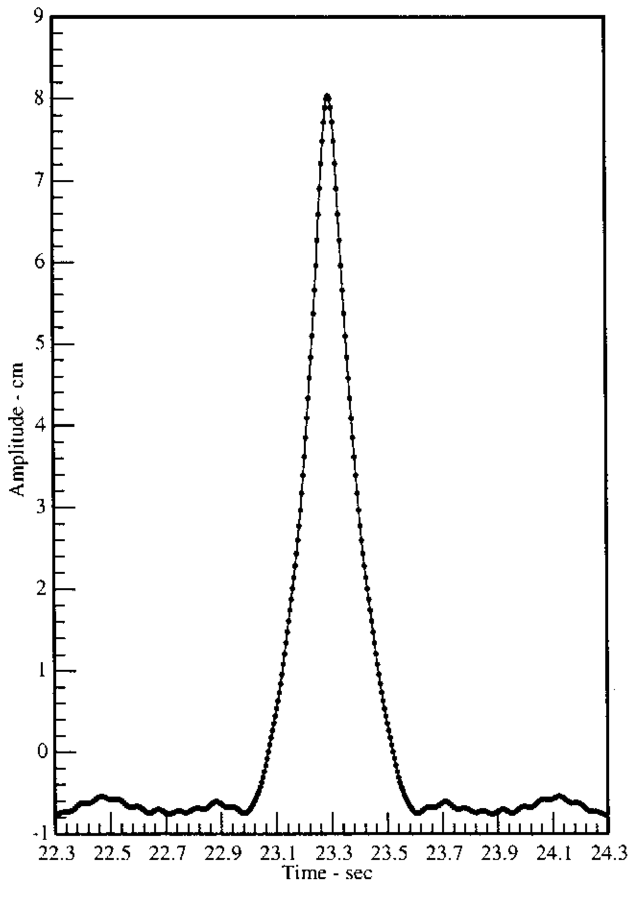

Fig. 6. Analysis of the "symmetric part" of a typical "almost highest wave" which was obtained from the uppermost signal in Fig. 5. This is a very nonlinear wave form, with an amplitude of 8.7 cm, in only 11.5 cm water depth with amplitude-to-depth ratio 0.76.

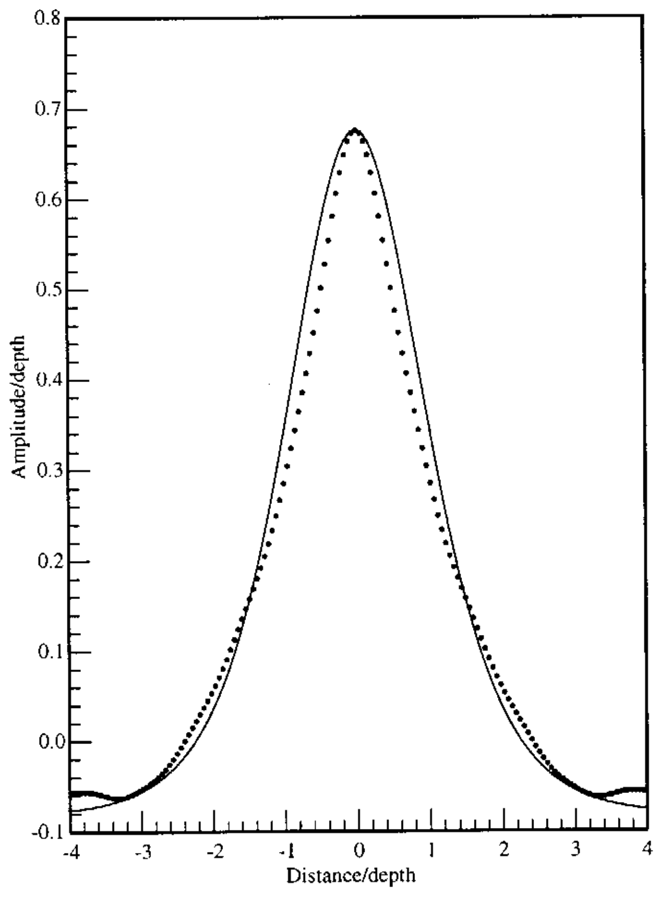

Fig. 7. Expanded view of the wave in Fig. 6 (dots). Also shown is a KdV solitary wave of the same amplitude (solid line). These results suggest that KdV dynamics are not very precise for high waves in shallow water.

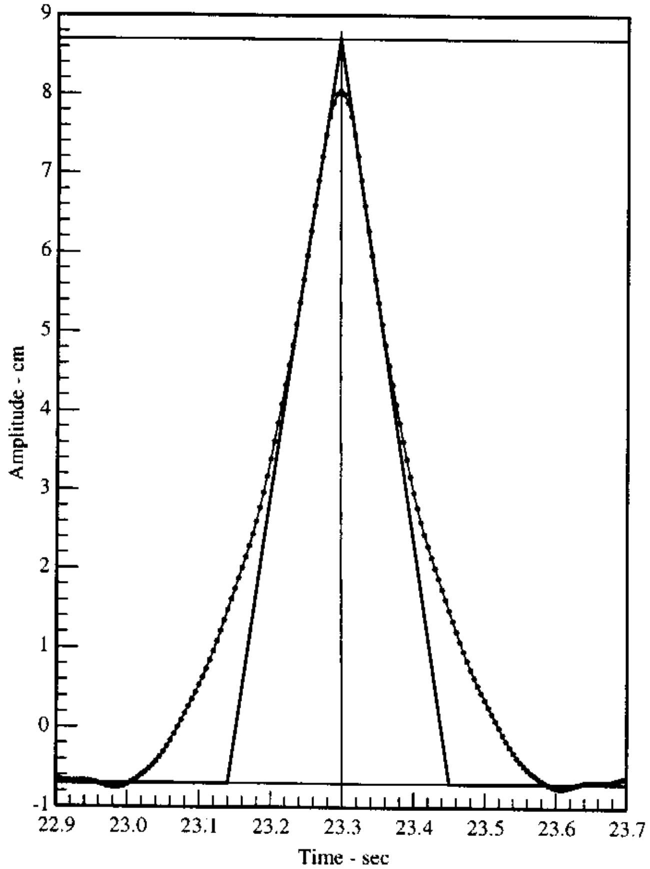

Fig. 8. Expanded view of the solitary wave in Fig. 6. Here an angle of 120° has been superposed over the measured wave near the peak. The results suggest that the measured waves have a peak angle near 120°, except for the region nearest the peak where natural excitations of a hydrodynamic instability might contribute to erosion of the crest of the wave.

wave (soliton) solution of the KdV equation? The intuitive feeling that one gets by observing the waves as they propagate from deep water into shallower water, based purely on Fig. 5, is that the waves undergo substantial nonlinear evolution. But, once again, is this evolution very different from KdV evolution? In Fig. 7 I compare the peaked wave of Fig. 6 with the KdV solitary wave solution (the amplitude was taken to be that of the measured wave (8.7 cm) for which the local depth was 11.5 cm). On the basis of Fig. 7 the measured wave form is seen to be substantially different than the KdV solitary wave, a result which suggests that higher order nonlinear effects influence the measured dynamics.

It is not easy to quantitatively state whether the measured waves correspond to the "highest waves" found theoretically by Stokes. Further experiments will refine this search, but for the present we may use the uppermost signal of Fig. 5 as a good representation of the most nonlinear waves measured in the experiments. In Fig. 8 I superpose an angle of 120° over the peaked wave form of Fig. 6; the angle was shifted vertically as shown, to be consistent with the analysis of Section 8 below. The aim is to see whether the peak angle predicted by Stokes for the highest wave is in any way represented by the data. It can be seen that the agreement between the wave form and the angle is rather good, except very near the peak itself.

There are several possible reasons for this lack of agreement. The first is that there is a *physical instability* which occurs because the acceleration of a typical water parcel near the peak is approximately equal to the gravitational acceleration, g (Kinsman, 1965; Tanaka, 1986). Thus there are only small residual forces to keep the water parcels on the free surface near the crest. Consequently, for a small region near the peak of the highest wave, the free surface is subject to disruption from external forces of all kinds which are not normally included in the constant-depth, unidirectional water wave equations. These effects include the presence of dissipation in the fluid boundary layer on the bottom and the sides of the canal, the influence of air currents, the possible presence of mechanical vibrations in the canal, the reflections of waves from the bottom slope as the waves propagate into shallow water, etc. Another possible influence, often excluded in considerations with regard to the highest wave, is the possible generation of capillary waves and their resultant excitations which may influence the large waves. Consequently there are several external forces which could excite the instability at the crest of the solitary wave and which could possibly erase evidence of the 120° Stokes peak angle. These and other considerations would suggest that the 120° angle is a rather delicate result, particularly for physical waves subjected to real world conditions. On the basis of these experimental observations (Fig. 8) it appears plausible that ~ 0.6 cm of the wave height might have been eroded away due to this instability.

Another reason why one might expect to lose information about the behavior of the largest waves near the peak is the fact that experimental uncertainties might occur due to possible *randomness in the phase* of the waves arriving at the last probe. These uncertainties could possibly arise from errors in the control and feedback loop used to generate the wave trains. The influence of these errors has yet to be fully explored.

In consequence of these simple experiments it seems reasonable to conclude that higher order theories might be exploited to investigate some of the highly nonlinear dynamics of shallow water waves at large Ursell numbers. Many of these issues are explored theoretically and numerically in the rest of this paper.

From a (potential flow) theoretical point of view the evolution of a sine wave as it propagates into a shallow water region (with small bottom slope) results in a slow, adiabatic deformation of the wave shape from the sine wave itself to a Stokes wave to a solitary wave (the soliton solution to the KdV equation) to a narrower and higher solitary wave up to the highest wave studied by Stokes. This adiabatically varying periodic wave train corresponds approximately, at any particular instant, to a travelling wave solution of the water wave equations. As discussed below in Section 9 the study of the nonlinear evolution of a sine wave in decreasing depth reveals many of the physical properties of a *single degree of freedom* solution of the nonlinear water wave equations. From the point of view of the inverse scattering transform this travelling wave form is a *basis function* of the nonlinear Fourier transform of a higher order wave equation. These waves depend strictly on two parameters, the amplitude-to-depth ratio and the depth-to-wave-length ratio. When these small parameters change due to propagation up a linear slope the travelling wave deforms as shown in the experiments of Fig. 5. As discussed in Section 9 solutions of higher order wave equations with periodic boundary conditions consist of the linear superposition of these travelling waves plus interactions among them (Osborne, 1997). It is therefore quite important to fully characterize the travelling waves in the context of nonlinear Fourier analysis; this effort requires exploration of higher order, integrable wave theories as discussed in the following Sections.

4 Multiscale Expansion of the Shallow Water Wave Equations for Unidirectional Motion

In this Section I follow Whitham and summarize his formal expansion of the equations of motion for unidirectional, shallow water wave motion (Whitham,

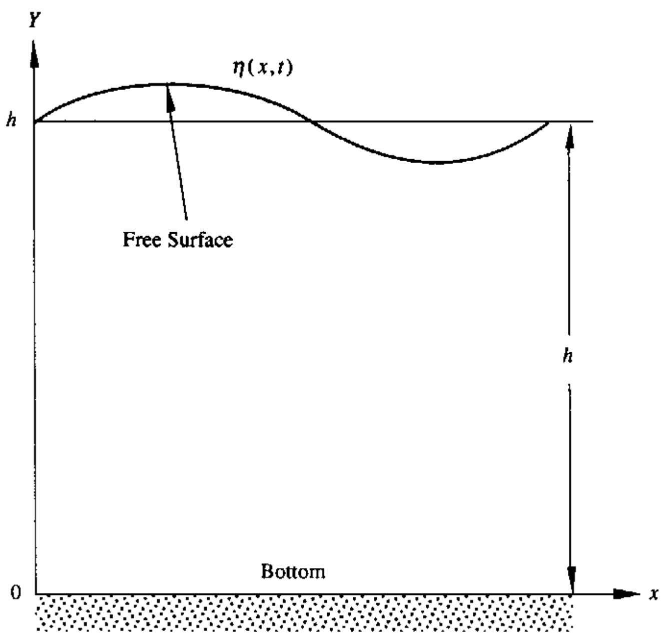

Fig. 9. Definition sketch of the coordinate frame used herein for shallow water wave motion.

1974). It is convenient, but not necessary, to assume that the fluid is irrotational and to thereby introduce a velocity potential. The domain of the wave equations is rectangular, (x, Y), where x is the horizontal coordinate and Y is measured from the bottom up to the small-but-finite-amplitude free surface, $h+\eta(x,t)$ (see Fig. 9); $\eta(x,t)$ is the wave amplitude relative to the undisturbed free surface. In this domain one seeks to solve Laplace's equation for the velocity potential ϕ:

$$\phi_{xx}+\phi_{YY}=0, \qquad 0<Y<h+\eta \tag{16}$$

where $\phi_Y=0$ on $Y=0$ (i.e. the vertical velocity component is zero on the bottom). It is physically intuitive that for shallow water waves the horizontal velocity, ϕ_x, should depend (roughly) linearly on the height of the waves (recall that for KdV ϕ_x is proportional to the wave amplitude $\eta(x,t)$). Thus one might generally expect an expansion for the velocity potential which depends on Y in the form of a simple power series:

$$\phi=\sum_{n=0}^{\infty} Y^n f_n(x,t) \tag{17}$$

an expression which suggests possible rapid converge in the shallow water approximation. Substituting (17) into the Laplace equation (16) and applying the bottom boundary condition one finds

$$\phi=\sum_{m=0}^{\infty}(-1)^m \frac{Y^{2m}}{(2m)!}\frac{\partial^{2m} f}{\partial x^{2m}} \tag{18}$$

where $f=f_o$.

In order to proceed further one needs the full Euler equations, including the nonlinear surface boundary conditions. The original variables in the problem are normalized in the following way:

$$x=lx', \qquad Y=hY', \qquad t=\frac{lt'}{c_o}$$
$$\eta=a\eta', \qquad \phi=\frac{gla}{c_o}\phi' \tag{19}$$

Here l is a length scale and a is a characteristic wave amplitude. The small parameters $\alpha=a/h$ and $\beta=(h/l)^2$ are used in the multiscale expansion which follows. The *water wave equations* in normalized form are thus given by:

$$\beta\phi_{xx}+\phi_{YY}=0, \qquad 0<Y<1+\alpha\eta \tag{20a}$$

$$\phi_Y=0, \qquad Y=0 \tag{20b}$$

$$\left.\begin{aligned} &\eta_t + \alpha\phi_x\eta_x - \frac{1}{\beta}\phi_Y = 0 \\ &\eta + \phi_t + \frac{1}{2}\alpha\phi_x^2 + \frac{1}{2}\frac{\alpha}{\beta}\phi_Y^2 = 0 \end{aligned}\right\} \quad Y = 1 + \alpha\eta \tag{20c}$$

The primes have been dropped in the final scaled equations. The last two equations in (20) are the kinematic and dynamic surface boundary conditions respectively. The goal is to make a multiscale expansion of (20) in terms of the parameters α, β. Here I take $O(\alpha) \sim O(\beta)$ (Kruskal's principle of maximum balance), but this may be neither necessary nor advisable in other circumstances. In this way one finds the hierarchy of equations, Wn, to arbitrary order n.

The solution to Laplace's equation now has the form

$$\phi = \sum_{m=0}^{\infty} (-\beta)^m \frac{(1+\alpha\eta)^{2m}}{(2m)!} \frac{\partial^{2m} f}{\partial x^{2m}} \tag{21}$$

which is essentially an expansion in powers of the small parameter β. Substituting (21) into the surface boundary conditions in (20) (the third and forth equations) gives:

$$\begin{aligned} &\eta_t + w_x + \alpha(\eta w)_x - \tfrac{1}{6}\beta w_{xxx} - \tfrac{1}{2}\alpha\beta(\eta w_{xx})_x + \\ &\qquad -\tfrac{1}{2}\alpha^2\beta(\eta^2 w_{xx})_x + O(\beta^2) = 0 \\ &w_t + \eta_x + \alpha w w_x + \tfrac{1}{2}\beta\eta_{xxx} + \alpha\beta(\eta\eta_{xx} + w_x^2)_x + \\ &\qquad +\alpha^2\beta(2\eta w_x^2 + \tfrac{1}{2}\eta^2\eta_{xx})_x + O(\beta^2) = 0 \end{aligned} \tag{22}$$

for $w = f_x$. Assume at this point that $O(\alpha) \sim O(\beta)$, and, without loss of generality, let $\alpha = 2\varepsilon/3$ and $\beta = 6\varepsilon$ (with Fokas and Liu, 1996) to find at $O(\varepsilon^2)$:

$$\begin{aligned} &\eta_t + w_x + \tfrac{2}{3}\varepsilon(\eta w)_x - \varepsilon w_{xxx} + O(\varepsilon^2) = 0 \\ &w_t + \eta_x + \tfrac{2}{3}\varepsilon w w_x - 3\varepsilon w_{xxt} + O(\varepsilon^2) = 0 \end{aligned} \tag{23}$$

Assuming further that the waves are unidirectional one has:

$$\begin{aligned} &\eta_t + \eta_x + \varepsilon K(\eta) + \\ &\quad +\varepsilon^2(\alpha_1\eta_{xxxxx} + \alpha_2\eta\eta_{xxx} + \alpha_3\eta_x\eta_{xx} + \alpha_4\eta^2\eta_x) + \\ &\quad + O(\varepsilon^3) = 0 \end{aligned} \tag{24}$$

where the constant coefficients are $\alpha_1 = 19/10$, $\alpha_2 = 10$, $\alpha_3 = 23$ and $\alpha_4 = -6$ and

$$K(\eta) = \eta_{xxx} + 6\eta\eta_x \tag{25}$$

Note that the linear phase speed has the value $c_o = 1$ in the present normalization of the variables in (24).

By moving to the obvious Galilean frame (propagating with speed $c_o = 1$) and rescaling x and t (24) becomes:

$$\begin{aligned} &\eta_t + K(\eta) + \\ &\quad +\varepsilon(\alpha_1\eta_{xxxxx} + \alpha_2\eta\eta_{xxx} + \alpha_3\eta_x\eta_{xx} + \alpha_4\eta^2\eta_x) + \\ &\quad + O(\varepsilon^2) = 0 \end{aligned}$$

which is W2, i.e. KdV plus $O(\varepsilon)$ corrections (3a). Note that in the limit $\varepsilon \to 0$ one gets the KdV equation. This asymptotic property of the KdV equation was pointed out many years ago by Martin Kruskal, i.e. that KdV describes the generic nonlinear wave motion as $\varepsilon \to 0$. By extending the above multiscale expansion to $O(\varepsilon^2)$ one obtains W3, which is not addressed further herein.

5 Establishment of Integrability for Higher Order Equations

Fokas and Liu (1996) have recently established the asymptotic integrability of certain higher order wave equations. I address two of their "propositions" which are relevant in the present case for the possible integration of W2 (3a, b), namely:

Proposition 1: Let v solve the KdV equation (2), $v_t + K(v) = 0$. Let

$$u = v + \varepsilon[\lambda_1 v^2 + \lambda_2 v_{xx} + \lambda_3 v_x \partial^{-1} v + \lambda_4 x K(v)] \tag{26}$$

Here ∂^{-1} indicates integration with respect to x, $\lambda_1 = 118/57$, $\lambda_2 = 44/57$, $\lambda_3 = 52/57$ and $\lambda_4 = -1/3$. Then u solves W2 (3a).

Proposition 2: Let v solve the Camassa-Holm equation:

$$v_t + 6vv_x + v_{xxx} - \varepsilon(v_{xxt} + 2vv_{xxx} + 4v_x v_{xx}) = 0 \tag{27}$$

Let

$$u = v + \varepsilon[\lambda_1 v^2 + \lambda_2 v_{xx} + \lambda_3 v_x \partial^{-1} v] \tag{28}$$

where $\lambda_1 = 14/19$, $\lambda_2 = 2/19$ and $\lambda_3 = -8/19$. Then u solves W2 (3a).

What can we conclude from these propositions? That by adding some (non trivial) function (see (26) or (28)) to the solution of an integrable wave equation (namely KdV (2) or CH (27)), we solve a *higher order equation* W2 (3a) to $O(\varepsilon)$. Note that in both cases (3a) is solvable in the sense that the simple transformations (26), (28) lead to higher order terms which are neglected in the analysis, i.e. they occur at $O(\varepsilon^2)$ and are hence not included in the specific $O(\varepsilon)$ terms given in (3a).

It is also worth noting that the time derivative which appears in (27) at $O(\varepsilon)$ is "deregularized" when applying the transformation (28) by using the leading order behavior governed by the KdV equation: $u_t \cong -(6uu_x + u_{xxx})$. These ideas are important for the integration of higher order equations as discussed in Section 6.

The above propositions are based upon the following observation (Fokas and Liu, 1996). Let v solve the following wave equation:

$$v_t + M_0(v) + \varepsilon \hat{M}_1(v) = 0 \tag{29}$$

Let u be defined by

$$u = v + \varepsilon P(v) \tag{30}$$

Then u solves

$$u_t + M_0(u) + \varepsilon \hat{M}_1(u) + \varepsilon [P, M_0]_L(u) + O(\varepsilon^2) = 0 \tag{31}$$

where the commutator $[\,.,.\,]_L$ is defined by

$$[A,B]_L = A'B - B'A$$
$$A' = \frac{\partial A}{\partial u} + \frac{\partial A}{\partial u_x}\partial_x + \frac{\partial A}{\partial u_{xx}}\partial_x^2 + \ldots \tag{32}$$

We thus have a formal way to transform from an integrable equation, $v_t + M_0(v) + \varepsilon \hat{M}_1(v) = 0$, to some other integrable equation (31) via a simple transformation (30). Depending upon the choice for the function $P(v)$ one obtains a specific integrable equation. As suggested by Fokas and Liu (1996) an appropriate choice for the function $P(v)$ is the *master symmetry* of the integrable equation, $v_t + M_0(v) = 0$, where the master symmetry is defined by $[P, M_0]_L = \hat{M}_1$, such that $\hat{M}_1$ is the next commuting flow of the hierarchy of integrable equations (Fuchsteiner and Fokas, 1981).

For the particular choice of the KdV equation we have $M_0 = K$ and $\hat{M}_1 = 0$, for which one has

$$P = \lambda_1 v^2 + \lambda_2 v_{xx} + \lambda_3 v_x \partial^{-1} v + \lambda_4 x K(v) \tag{33}$$

Then the equation of motion becomes

$$u_t + K(u) + \varepsilon K_1(u) = 0 \tag{34}$$

For the particular choices $\lambda_1 = 8/3$, $\lambda_2 = 4/3$, $\lambda_3 = 2/3$ and $\lambda_4 = 1/3$, one has

$$K_1(u) = u_{5x} + 10uu_{xxx} + 20v_x v_{xx} + 30u^2 u_x \tag{35}$$

which gives the integrable equation found by Kodama (4). This equation bears some resemblance to W2 (3b), but differs in the values of the constant coefficients. In order to arrive at the results of Propositions 1 and 2 it is natural to let the coefficients λ_i in (26), (28) be arbitrary and to hence compute the corresponding $\alpha_i = \alpha_i(\lambda_i)$ to agree with the values in W2. In this way one selects particular values of the λ_i in (33) to give the proper coefficients α_i for W2 (3b). The appropriate values of the λ_i are given Proposition 1.

In Proposition 2 one has the CH equation such that $M_0 = K$ and

$$\hat{M}_1 = -(v_{xxt} + 2vv_{xxx} + 4v_x v_{xx})$$

where $P = \lambda_1 v^2 + \lambda_2 v_{xx} + \lambda_3 v_x \partial^{-1} v$. The time derivative in $\hat{M}_1$ is deregularized using the KdV equation in all subsequent calculations. Again the coefficients λ_i are adjusted appropriately as before and are found to be those given in Proposition 2.

We are now lead to the following question: Can W2 (3a, b) be integrated *exactly*, or even *approximately*, by the inverse scattering transform? This is a nontrivial question, since asserting integrability (as was done above) is quite different from the process of carrying out the integration itself. An attempt to address an approach leading to the approximate integration of W2 is the focus of the next Section.

6 Systematic Approach for the Integration of Higher Order Equations

This Section is dedicated to describing a series of ideas leading to the concept of *universal Lax pairs* and their associated *universal nonlinear wave equations*. Here the term "universal" indicates that the formulation depends upon an arbitrary function which can be appropriately selected to integrate, or approximately integrate, a particular higher order wave equation.

I first discuss integrability for the KdV equation and then give a generalization of the requisite Lax pair for KdV, which is subsequently applied to integrate the CH equation (27). On this basis I then formulate the universal Lax pair ((51) below) and apply it to the approximate integration of W2 (3a, b).

6.1 The Korteweg-deVries Equation

The KdV equation (2) is integrable by the Schroedinger eigenvalue problem

$$\psi_{xx} + [u(x,0) + E]\psi = 0 \tag{36}$$

($E \sim k^2$ is the eigenvalue and k is the wave number) together with the time dependence for the eigenfunction

$$\psi_t = a(u;E)\psi - b(u;E)\psi_x \tag{37}$$

where $b = 2u - 4E$. Equations (36), (37) are referred to as a *Lax pair* (Lax, 1968; Ablowitz and Segur, 1981; Calogero and Degasperis, 1982; Dodd, et. al, 1982; Novikov, et al, 1984; Newell, 1985; Drazin and Johnson, 1989). The compatibility condition $\psi_{xxt} = \psi_{txx}$ yields the KdV equation, which is of course independent of the spectral parameter, E, a requirement for integrability (it further follows from the compatibility condition that $a \sim b_x / 2 \sim u_x$).

6.2 Generalization of the Eigenvalue Problem

Is there some way to extend the applicability of the spectral problem (36) to other physically interesting equations? To this end I suggest a generalized eigenvalue problem

$$\psi_{xx} + Q(x;E)\psi = 0 \tag{38}$$

where $Q(x;E)$ is specified in (40) and (45) below for the problems of interest herein. Applying the compatibility condition to the Lax pair (37), (38) gives the *nonlinear wave equation*:

$$Q_t + 2b_x Q + bQ_x + \tfrac{1}{2} b_{xxx} = 0 \tag{39}$$

In the computation of this equation one eliminates $a(E)$ via the derived relation: $a_{xx} = b_{xxx} / 2$. Should (39) be independent of the eigenvalue E then the Lax pair (37), (38) is said to *integrate* (39) and the full mathematical machinery of the inverse scattering transform can be brought to bear in order to understand the physical behavior of (39) and to compute its solutions for various selected boundary conditions. Thus, specification of appropriate functions for $Q(u, u_x \ldots, E)$ in (38) and $b(u, u_x \ldots, E)$ in (37) could possibly lead to an integrable wave equation of physical interest for (39). The functions u and Q are here assumed to be well-behaved differentiable functions which go to zero sufficiently fast as $|x| \to \infty$ on the infinite line ($-\infty < x < \infty$), or are periodic ($u(x,t) = u(x+L,t)$) on the interval $0 \le x \le L$.

6.3 The Camassa-Holm Equation

Consider a transformation of the KdV Lax pair ((37), (38)) to a new variable, $u \to u + \varepsilon P(u)$, where $P(u)$ is to be chosen below and ε is a small parameter. In order to search for the associated wave equation arising from this transformation one might chose as a candidate for $Q(u, u_x \ldots, E)$ in (38):

$$Q = \alpha(E) m(x,0) + \gamma(E) \tag{40}$$

where $m = u + \varepsilon P(u)$ with associated time-dependence coefficient $b = 2u - \beta(E)$. Note that this procedure is similar to that for the KdV equation, but is modified by including additional possible spectral dependencies in the functions $\alpha(E)$, $\beta(E)$ and $\gamma(E)$ (for $\varepsilon \to 0$ the results reduce to those for the KdV equation and the above functions are 1, $4E$, E, respectively). Use (40) and $b = 2u - \beta(E)$ in (39) to get:

$$\begin{aligned} &\alpha(E) = 1 + 4\varepsilon E \\ &\beta(E) = \frac{4\gamma(E)}{\alpha(E)} = \frac{4E}{1+4\varepsilon E} \\ &\gamma(E) = E \\ &P(v) = -u_{xx} \end{aligned} \tag{41}$$

for which the wave equation is computed to be

$$m_t + 4mu_x + 2um_x + u_{xxx} = 0 \tag{42}$$

Note that $P(u) = -u_{xx}$ is the most obvious choice here, but not the only one (see discussion in Section 10 below). Using $m = u + \varepsilon P(u) = u - \varepsilon u_{xx}$ in (42) gives the familiar Camassa-Holm equation (27) (1993; Camassa et al., 1994):

$$u_t + 6uu_x + u_{xxx} = \varepsilon(u_{xxt} + 4u_x u_{xx} + 2uu_{xxx}) \tag{43}$$

with Lax pair

$$\begin{aligned} &\psi_{xx} + \{(1+4\varepsilon E)[u - \varepsilon u_{xx}] + E\}\psi = 0 \\ &\psi_t = u_x \psi - 2\left[u - \frac{2E}{1+4\varepsilon E}\right]\psi_x \end{aligned} \tag{44}$$

where, in appropriate variables, $\varepsilon = h^2 / 3$.

Thus by adding a particular function $P(u)$ to the solution of the KdV equation we obtain and easily integrate (by generalizing the Lax pair for the KdV equation) a well-known water wave equation (43) which has a physical basis (it arises in the study of the Green-Nagdy equations for small-but-finite-amplitude waves in shallow water (Camassa and Holm, 1993)). Letting $\varepsilon \to 0$ in (43) and (44) gives the KdV equation (2) and its Lax pair (36), (37). It is in this sense that the CH equation may be viewed as a natural extension of the KdV equation.

6.4 A Universal Wave Equation and Its Lax Pair

Can the procedure given above for adding an appropriate function ($-\varepsilon u_{xx}$) to an existing wave equation solution ($u(x,t)$ of the KdV equation) be applied once again in order to integrate other physical wave equations of interest?

To this end I extend (43), (44) by letting $u \to u + \mu N$, where μ is a small number which is $O(\varepsilon)$ (but for now it will be left distinct) and $N(x,t)$ is assumed, for concreteness, to be a well-behaved differentiable function which goes to zero sufficiently fast as $|x| \to \infty$ on the infinite line ($-\infty < x < \infty$) or is periodic ($N(x,t) = N(x+L,t)$) on the interval $0 \le x \le L$. Specifically the dependence $N = N(u, u_x, u_{xx}, \ldots)$ is assumed throughout. Hence, in analogy with (40), it is appropriate to define

$$Q = \alpha(E)M(x) + \gamma(E) \tag{45}$$

where

$$M(x) = u - \varepsilon u_{xx} + \mu(N - \varepsilon N_{xx}) \tag{46}$$

and to generalize the coefficient b in the time dependence of the eigenfunction in the obvious way:

$$b = 2u - \beta(E) + 2\mu N \tag{47}$$

Equations (45)-(47) together with (37), (38) constitute a possible candidate Lax pair for this problem. Use (45), (47) in (39) and find that the first three equations of (41) hold once again in the present case. The eigenvalue E is eliminated easily from the formulation and the following nonlinear wave equation arises (in the particular shorthand notation):

$$\boxed{M_t + C(M, u + \mu N) = 0} \tag{48}$$

where the operator $C(u,v)$ has the definition (see (42))

$$C(u,v) = 4uv_x + 2vu_x + v_{xxx} \tag{49}$$

It is useful to note that $C(u,v)$ has the properties:

$$\begin{aligned} &C(u, v+w) = C(u,v) + C(u,w) \\ &C(u+v, w) = C(u,w) + C(v,w) - w_{xxx} \\ &C(u, av) = aC(u,v) \\ &C(u,0) = 0; \qquad C(0,w) = w_{xxx} \end{aligned} \tag{50}$$

for a constant. This leads to an alternate form for (48)

$$M_t + C(M,u) + \mu C(M,N) = 0$$

Setting $\mu = 0$ gives the CH equation (42), $m_t + C(m,u) = 0$, in the operator notation of (49).

It follows that the equation of motion (48) is integrable by the inverse scattering transform using the following Lax pair (use (45), (46) in (38) and (47) in (37)):

$$\begin{gathered} \boxed{\psi_{xx} + \{(1+4\varepsilon E)[u - \varepsilon u_{xx} + \mu(N - \varepsilon N_{xx})] + E\}\psi = 0} \\ \boxed{\psi_t = [u_x + \mu N_x]\psi - 2\left[u + \mu N - \frac{2E}{1+4\varepsilon E}\right]\psi_x} \end{gathered} \tag{51}$$

Thus specification of *any* well-behaved function N in the Lax pair (51) yields an integrable wave equation (48). Stated differently, simply by adding an arbitrary function $\mu N(x,t)$ to the solution of the CH equation (43), and to its Lax pair (44), we obtain a new integrable wave equation (48) with Lax pair (51).

Can (48) also be a *physically important wave equation*? To this end note that as $\mu \to 0$ (48) reduces to the CH equation and if, further, $\varepsilon \to 0$ (48) reduces to KdV. Hence (48) may be viewed as a wave equation which is the third step in a procedure that leaps from KdV to CH to some other equation at the same and higher order.

A remarkable property of the new nonlinear wave equation (48) is that it is integrable for *arbitrary N*; it is for this reason that I refer to (48) as a *universal wave equation* and (51) as its *universal Lax pair*. By picking the function N one is able to integrate an infinite number of nonlinear wave equations. By picking the *correct function N* one might also be able to integrate certain *physical wave equations* of interest. Furthermore, by picking the *appropriate function N*, together with its specified parameters (which I typically call λ_i, $i = 1,2\ldots$) one can attempt to "tune" or to "match at $O(\varepsilon)$" the universal equation (48) to a wave equation such as (3b) in order to integrate or approximately integrate the nonlinear wave motion. This procedure is discussed in detail in the next Subsection.

Expanding (48) yields the new *integrable, universal wave equation* in terms of the arbitrary function N:

$$\begin{aligned} &u_t + 6uu_x + u_{xxx} - \varepsilon[u_{xxt} + 4u_x u_{xx} + 2uu_{xxx}] + \\ &\quad + \mu[N_t + 6(uN_x + u_x N) + N_{xxx}] + \\ &\quad - \mu\varepsilon[N_{xxt} + 4(N_{xx}u_x + u_{xx}N_x) + 2(Nu_{xxx} + uN_{xxx})] + \\ &\quad + 6\mu^2 NN_x - 2\mu^2\varepsilon[2N_x N_{xx} + NN_{xxx}] \end{aligned} \tag{52}$$

This is the CH equation plus $O(\mu)$, $O(\mu\varepsilon)$, $O(\mu^2)$ and $O(\mu^2\varepsilon)$ terms. Without loss of generality one can assume $\mu \sim \varepsilon$. Then (52) can be interpreted in terms of the CH equation plus specific $O(\varepsilon)$, $O(\varepsilon^2)$ and $O(\varepsilon^3)$ terms:

$$
\begin{aligned}
&u_t + 6uu_x + u_{xxx} + \\
&+\varepsilon[N_t + 6(uN_x + u_xN) + N_{xxx} - u_{xxt} - 4u_xu_{xx} - 2uu_{xxx}] \\
&-\varepsilon^2[N_{xxt} + 4(N_{xx}u_x + u_{xx}N_x) + 2(Nu_{3x} + uN_{3x}) - 6NN_x] \\
&-2\varepsilon^3[2N_xN_{xx} + NN_{xxx}]
\end{aligned}
\tag{53}
$$

Application of the compatibility condition to the Lax pair (51) directly gives the integrable equation (53). The first two lines in (53) can be interpreted as being related to particular *target equations* such as (3b); the remaining terms at $O(\varepsilon^2)$ and $O(\varepsilon^3)$ are "corrections" to (3b) which must be made in order to insure integrability by the Lax pair (51). In what follows the $O(\varepsilon)$ terms in (53) must be rendered "equivalent" to the $O(\varepsilon)$ terms in (3b). This consideration is addressed in the following Subsection.

6.5 Targeting and Matching the Second Equation in the Whitham Hierarchy (W2) to $O(\varepsilon)$

At this point one must presumably make a choice for the function $N(x,t)$ in order to obtain a concrete wave equation for (53). One could be tempted by the many interesting forms which might be selected for N (all infinity of them) but instead I use (28), on the basis of Proposition 2, to provide an appropriate choice:

$$N = \lambda_1 u^2 + \lambda_2 u_{xx} + \lambda_3 u_x \partial^{-1} u \tag{54}$$

The next step is to insert (54) into the $O(\varepsilon)$ terms in (53) to find the resultant wave equation:

$$
\begin{aligned}
&u_t + 6uu_x + u_{xxx} + \\
&\quad +\varepsilon[2\lambda_1 uu_t + (\lambda_2 - 1)u_{xxt} + \lambda_3 u_{xt}\partial^{-1}u + \lambda_3 u_x \partial^{-1} u_t + \\
&\qquad +\lambda_3 K_x(u)\partial^{-1}u + \lambda_2 u_{xxxxx} + \\
&\qquad +(2\lambda_1 + 6\lambda_2 + 3\lambda_3 - 2)uu_{xxx} + \\
&\qquad +2(3\lambda_1 + 3\lambda_2 + 2\lambda_3 - 2)u_xu_{xx} + \\
&\qquad +6(3\lambda_1 + \lambda_3)u^2u_x] + \\
&\qquad +O(\varepsilon^2, \varepsilon^3) = 0
\end{aligned}
\tag{55}
$$

Note that the terms $O(\varepsilon^2, \varepsilon^3)$ have been left out in this expression as they are not necessary to complete the targeting operation. These higher order terms are given in (59) below.

The next goal is to force the $O(\varepsilon)$ terms in (55) to match the $O(\varepsilon)$ terms in the target equation (3b). This operation thus insures the ***equivalence*** of (55) and (3b) to $O(\varepsilon)$. To do this one computes the α_i in (3b) in terms of the λ_i in (54). In order to determine the matching, the time derivatives in the $O(\varepsilon)$ terms in (55) are replaced by spatial derivatives using the leading order approximation from the KdV equation: $u_t \cong -(6uu_x + u_{xxx})$. The resultant *deregularized form* of (55) is then found to be:

$$
\begin{aligned}
&u_t + 6uu_x + u_{xxx} + \\
&\quad +\varepsilon[u_{xxxxx} + (3\lambda_3 + 4)uu_{xxx} + \\
&\qquad +(6\lambda_1 - 12\lambda_2 + 3\lambda_3 + 14)u_xu_{xx} + \\
&\qquad +3(2\lambda_1 + \lambda_3)u^2u_x - \mathcal{C}\lambda_3 u_x] + \\
&\qquad +O(\varepsilon^2, \varepsilon^3) = 0
\end{aligned}
\tag{56}
$$

where $\mathcal{C}$ is an integration constant; in the present application for surface waves it can be set to zero. However, for internal waves $\mathcal{C}$ has a finite physical value (Lee and Beardsley, 1974; Osborne, 1997). The expression (56) at $O(\varepsilon)$ can be used to provide equations for the $\alpha_i = \alpha_i(\lambda_i)$. This is done by comparing (56) with (3b):

$$
\begin{aligned}
\alpha_1 &= 1 \\
\alpha_2 &= 3\lambda_3 + 4 \\
\alpha_3 &= 6\lambda_1 - 12\lambda_2 + 3\lambda_3 + 14 \\
\alpha_4 &= 3(2\lambda_1 + \lambda_3)
\end{aligned}
\tag{57}
$$

Inverting these equations allows the coefficients λ_i in (54) to be computed in terms of the physical constants α_i in (3b):

$$
\begin{aligned}
\lambda_1 &= \tfrac{1}{6}(\alpha_4 - \alpha_2 + 4) \\
\lambda_2 &= \tfrac{1}{12}(\alpha_4 - \alpha_3 + 14) \\
\lambda_3 &= \tfrac{1}{3}(\alpha_2 - 4)
\end{aligned}
\tag{58}
$$

The inverted equations $\lambda_i = \lambda_i(\alpha_i)$ provide the precise coefficients in (54) to *match* equation (55) with the target equation (3b) to $O(\varepsilon)$. The resultant values of $\lambda_i = \lambda_i(\alpha_i)$ are then used in the $O(\varepsilon)$ and higher terms in (55) to insure integrability (see (59) below).

Note that for the values α_i in (3b) one finds from (58) the coefficients $\lambda_1 = -14/19$, $\lambda_2 = -2/19$ and $\lambda_3 = 8/19$. These values of λ_i have opposite signs relative to those found in Proposition 2 (28); this is because the transformation used here $(u \to u + \varepsilon P(u))$ for determining the universal Lax pair is equivalent to

$$u = v - \varepsilon[\lambda_1 v^2 + \lambda_2 v_{xx} + \lambda_3 v_x \partial^{-1} v]$$

rather than (28), i.e. there is effectively a minus sign in front of the ε. Thus the results given here are completely consistent with Fokas and Liu (1996).

In summary, one can view (55) in light of Section 5 and the procedure discussed for establishing integrability of W2 (3b). By discarding terms at $O(\varepsilon^2)$ and higher in (55) and by deregularizing the $O(\varepsilon)$ term one arrives at W2 (3b) with the appropriate coefficients given by (57). Thus (55) is an *extended and regularized version* of W2 which is a completely integrable wave equation (here referred to as exRW2, (59) below). The integrable equation (55), to $O(\varepsilon)$, is formally *equivalent* to (3b) to $O(\varepsilon)$. More precisely the integrable equation (55) contains particular regularized terms (e.g., with time derivatives, t, in place of certain space derivatives, x), plus terms at $O(\varepsilon^2)$ and $O(\varepsilon^3)$, which insure integrability. It is in this way that one integrates the higher order equations from the multiscale expansion of the water wave equations.

It should come as no surprise that (55) can be written in the form of Section 5:

$$u_t + K(u) + \varepsilon C(u) + \varepsilon[N,K]_L + O(\varepsilon^2, \varepsilon^3) = 0$$

where

$$K(u) = 6uu_x + u_{xxx}$$

$$C(u) = -(u_{xxt} + 4u_x u_{xx} + 2uu_{xxx})$$

and

$$\begin{aligned}[N,K]_L = {} & 2\lambda_1 uu_t + \lambda_2 u_{xxt} + \lambda_3 u_{xt}\partial^{-1}u + \lambda_3 u_x \partial^{-1} u_t + \\ & +\lambda_3 K_x(u)\partial^{-1}u + \lambda_2 u_{xxxxx} + \\ & +(2\lambda_1 + 6\lambda_2 + 3\lambda_3)uu_{xxx} + \\ & +2(3\lambda_1 + 3\lambda_2 + 2\lambda_3)u_x u_{xx} + \\ & +6(3\lambda_1 + \lambda_3)u^2 u_x]\end{aligned}$$

It is not hard to show that the commutator $[N,K]_L$ is given by this explicit form (with an obvious extended definition to include time derivatives in the formulation); to match this result to W2 one must of course deregularize the equation as previously discussed.

Finally, I note that for a particular choice of the parameters in the master symmetry (54), $\lambda_1 = 4$, $\lambda_2 = 2$ and $\lambda_3 = 2$, one gets the integrable Kodama equation (4). Likewise from (7) we have for the deregularized CH equation that $\alpha_1 = 1$, $\alpha_2 = 4$, $\alpha_3 = 14$ and $\alpha_4 = 0$ so that (58) gives $\lambda_1 = \lambda_2 = \lambda_3 = 0$, as expected. This operation is equivalent to setting the function $N(x) = 0$ by instead setting all the parameters λ_i equal to zero in (54).

For completeness I give the final asymptotically matched equation for (55); here the appropriate values of the $\lambda_i = \lambda_i(\alpha_i)$ (58) are assumed ($\lambda_1 = -14/19$, $\lambda_2 = -2/19$ and $\lambda_3 = 8/19$):

$$\begin{aligned} & u_t + 6uu_x + u_{xxx} + \\ & +\varepsilon[2\lambda_1 uu_t + (\lambda_2 - 1)u_{xxt} + \lambda_3 u_{xt}\partial^{-1}u + \lambda_3 u_x \partial^{-1}u_t + \\ & \quad +\lambda_3 K_x(u)\partial^{-1}u + \lambda_2 u_{xxxxx} + \\ & \quad +(2\lambda_1 + 6\lambda_2 + 3\lambda_3 - 2)uu_{xxx} + \\ & \quad +2(3\lambda_1 + 3\lambda_2 + 2\lambda_3 - 2)u_x u_{xx} + \\ & \quad +6(3\lambda_1 + \lambda_3)u^2 u_x] + \\ & -\varepsilon^2\{2(\lambda_1 + \lambda_3)uu_{xxt} + 2(\lambda_1 + \lambda_3)u_t u_{xx} + 2(2\lambda_1 + \lambda_3)u_x u_{xt} \\ & \quad +\lambda_2 u_{4xt} + \lambda_3 u_{3xt}\partial^{-1}u + \lambda_3 u_{3x}\partial^{-1}u_t \\ & \quad +[28\lambda_1 + 20\lambda_3 - 6\lambda_2(2\lambda_1 + \lambda_3)]uu_x u_{xx} + \\ & \quad +6(\lambda_1 + \lambda_3 - \lambda_1\lambda_2)u^2 u_{3x} + 4(2\lambda_1 + \lambda_3)u_x^3 + \\ & \quad +4\lambda_2 u_x u_{4x} + 2\lambda_2 uu_{5x} - 6\lambda_1(2\lambda_1 + \lambda_3)u^3 u_x + \\ & \quad -6\lambda_3^2 u_x u_{xx}(\partial^{-1}u)^2 + \qquad (59) \\ & \quad +[6\lambda_3(1 - \lambda_2)u_x u_{3x} + 2\lambda_3(2 - 3\lambda_2)u_{xx}^2 + 2\lambda_3 uu_{4x} + \\ & \quad -6\lambda_1\lambda_3 u^2 u_{xx} - 6\lambda_3(2\lambda_1 + \lambda_3)uu_x^2]\partial^{-1}u\} \\ & -\varepsilon^3\{(28\lambda_1^2 + 32\lambda_1\lambda_3 + 8\lambda_3^2)u^2 u_x u_{xx} + \\ & \quad +2\lambda_2(6\lambda_1 + 7\lambda_3)uu_{xx}u_{xxx} + 2\lambda_1(2\lambda_1 + 3\lambda_3)u^3 u_{3x} + \\ & \quad +4\lambda_2(3\lambda_1 + 2\lambda_3)u_x u_{xx}^2 + 2\lambda_1\lambda_2 u^2 u_{5x} + \end{aligned}$$

$$+2\lambda_2^2 u_{xx}u_{5x} + 4(2\lambda_1+\lambda_3)^2 uu_x^3 +$$

$$4\lambda_2(2\lambda_1+\lambda_3)u_x^2u_{xxx} + 4\lambda_2(2\lambda_1+\lambda_3)uu_xu_{4x} +$$

$$+4\lambda_2^2 u_{3x}u_{4x} +$$

$$+[8\lambda_3(\lambda_1+\lambda_3)uu_{xx}^2 + 4\lambda_3(5\lambda_1+3\lambda_3)u_x^2u_{xx} +$$

$$+4\lambda_2\lambda_3u_{xx}u_{4x} + 2\lambda_3(6\lambda_1+5\lambda_3)uu_xu_{xxx} +$$

$$+4\lambda_2\lambda_3u_{3x}^2 + 2\lambda_2\lambda_3u_xu_{5x} +$$

$$+2\lambda_1\lambda_3u^2u_{4x} + 2\lambda_2\lambda_3u_{xx}u_{4x}]\partial^{-1}u +$$

$$+(4\lambda_3^2u_{xx}u_{3x} + 2\lambda_3^2u_xu_{4x})(\partial^{-1}u)^2\}$$

This is the new nonlinear wave equation, exRW2, found and integrated herein. It is the main result of this paper. It has 52 terms and is found by evaluating (48) using the master symmetry (54) for particular coefficients λ_i found in the targeting operation of (3b). Eq. (59) illustrates the power of the new approach introduced herein, the *method of universal Lax pairs*, for integrating higher order wave equations. How else could one integrate a wave equation with 52 terms?

6.6 Other Interesting Nonlinear, Integrable Wave Equations

In the above results I have targeted the higher order wave equation W2 and thereby neglected other nearby, integrable wave equations. It is straightforward to take only certain terms in the master symmetry (54) and to arrive at other wave equations which have interesting structure. Some of these results are given in Table 2.

In the left column I show only the coefficients λ_i for which *finite values were chosen* and, for simply chosen numerical values of the λ_i, I give the coefficients of the terms in the resultant deregularized wave equations, α_i, in the remaining columns. Note, for example, that if *only* the coefficient λ_1 is retained (which means only the u^2u_x term in the master symmetry (54) is used) one finds an equation with coefficients 1, 4, 11 and -3, results which are not very far from W2 (which has approximate coefficients 1, 5.3, 12.1, -3.2). Formulas (57) give the α_i coefficient values in Table 2 corresponding to this case: $\lambda_1=-1/2$ and $\lambda_2=\lambda_3=0$. The simple resultant wave equation is then found by choosing λ_1 finite (and $\lambda_2=\lambda_3=0$) in (55):

$$u_t + 6uu_x + u_{xxx} +$$

$$+\varepsilon[2\lambda_1uu_t - u_{xxt} + 2(\lambda_1-1)uu_{xxx} +$$

$$+2(3\lambda_1-2)u_xu_{xx} + 18\lambda_1u^2u_x] + O(\varepsilon^2) = 0$$

The linear dispersion relation for this equation is identical with that for the CH equation, (64) below. This new equation is surprisingly simple and resembles the regularized equation W2 (55). I have not given the higher order terms here, but when appropriately computed they render the equation integrable. One can of course build up any number of integrable equations in this way as shown in Table 2.

α_i	u_{5x}	uu_{xxx}	u_xu_{xx}	u^2u_x
Kodama	1	10	20	30
CH	1	4	14	0
W2	1	-5.3	-12.1	-3.2
λ_1	1	4	11	-3
λ_2	1	4	12	0
λ_3	1	1	11	-3
λ_1, λ_2	1	4	12	-3
λ_1, λ_3	1	5	11	-3
λ_2, λ_3	1	1	15	-3

Table 2. Comparison of the constant coefficients α_i for each term in the Kodama (4), the deregularized CH (7) and the W2 (3b) equations, together with a number of simpler equations obtained using only certain choices for the terms in the master symmetry (54). In the left-hand column particular coefficients λ_i are given and are assumed to have certain finite values; those not specified are assumed to be zero.

7 Linear Dispersion Relation of the New Equation exRW2

The universal wave equation (59) is 'regularized' in the sense that certain space derivatives are expressed as time derivatives. This is an automatic consequence of the universal Lax pair (51), i.e. behavior of this type is guaranteed by application of the compatibility condition, $\psi_{xxt}=\psi_{txx}$. Therefore, with regard to the discussion in Section 1, it is not surprising to find that the linear dispersion relation for (59) is well behaved. The explicit form for the linearized exRW2 equation is found to be:

$$u_t + u_{xxx} + \varepsilon[(\lambda_2-1)u_{xxt} + \lambda_2u_{5x}] - \varepsilon^2\lambda_2u_{4xt} = 0 \quad (60)$$

Note that as $\lambda_2\to 0$ this equation reduces to the linearized CH equation ((64) below); furthermore as $\varepsilon\to 0$ we get the linearized KdV equation ($u_t+u_{xxx}=0$). To transform (60) into the laboratory

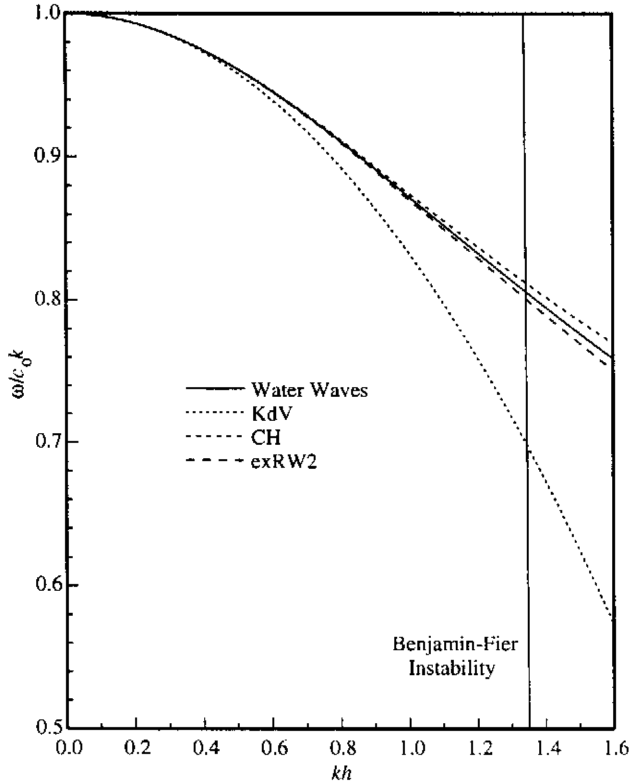

Fig. 10. Comparison of the linear dispersion relation for the linearized water wave equations with those for the linearized KdV, the CH and the exRW2 equations.

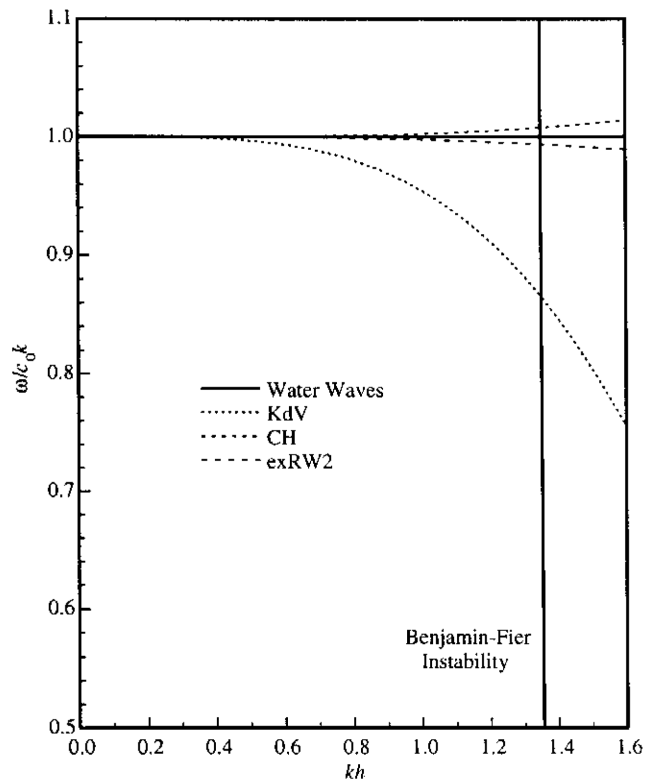

Fig. 11. Comparison of the dispersion relations for the KdV, the CH and the exRW2 equations *relative* to the water wave dispersion relation.

coordinate frame in dimensional units (η, x', t') the following expressions are applied:

$$\eta = u/\lambda$$

$$t = t'/\beta \tag{61}$$

$$x = x' + c_o t$$

for $\lambda = \alpha/6\beta$, $\alpha = 3c_o/2h$ and $\beta = c_o h^2/6$. Subsequently the primes are dropped and the linearized equation of motion for the exRW2 equation in the laboratory frame, in dimensional units, is then given by:

$$\begin{aligned}\eta_t + c_o\eta_x + \beta(2\lambda_2 - 1)\eta_{xxx} - \varepsilon(1-\lambda_2)\eta_{xxt} - \\ -\varepsilon\lambda_2\beta\eta_{5x} - \varepsilon^2\lambda_2\eta_{4xt} = 0\end{aligned} \tag{62}$$

The dispersion relation for this linearized equation then follows:

$$\omega = c_o k\left[\frac{1+(1-2\lambda_2)h^2k^2/6}{1+(1-\lambda_2)h^2k^2/3 - \lambda_2 h^4k^4/4}\right] \tag{63}$$

where $\lambda_2 = -2/19$.

These results should be contrasted to those for the linearized CH equation:

$$u_t + u_{xxx} = \varepsilon u_{xxt} \tag{64}$$

which has the dimensional form

$$\eta_t + c_o\eta_x - \beta\eta_{xxx} - \varepsilon\eta_{xxt} = 0 \tag{65}$$

with the dispersion relation

$$\omega = c_o k\left[\frac{1+h^2k^2/6}{1+h^2k^2/3}\right] \tag{66}$$

Note that the linear dispersion relation (63) reduces to (66) in the limit $\lambda_2 \to 0$.

A graph of the linear dispersion relations for the KdV, CH, exRW2 equations and the water wave equations, $\omega = \sqrt{gk}\tanh kh$, is shown in Fig. 10. Strictly speaking these results should be graphed only out to the Benjamin-Fier instability, $kh = 1.36$, because wave motion to the right of this value is modulationally unstable. Note that KdV dispersion falls substantially below that for the other three. Furthermore CH dispersion is greater than that for the linearized water wave equations at high values of kh, while the dispersion for the exRW2 equation is somewhat below that for the water wave equations. It is clear that the dispersion for exRW2 is more nearly in agreement with that for linearized water waves than are the other cases considered here.

In order to bring out the *relative* dispersion of the four cases I show in Fig. 11 the ratios of KdV, CH and exRW2 dispersion relative to that for linearized water waves, i.e. the function $\omega / \sqrt{gh \tanh kh}$ is presented for all four cases. In this graph the water wave relative dispersion is 1 for all values of kh. CH dispersion is somewhat above this value, exRW2 dispersion is slightly below and KdV is substantially below.

8 Solitary Wave of the New Equation exRW2

In this Section the focus is on the computation of the solitary wave solution of exRW2 (59), the solitary wave for the CH equation, the highest wave for pure potential flow (Evans and Ford, 1994) and the comparison of all three with laboratory data. In order to obtain the solitary wave of (59) (obviously not a simple task due to the large number of terms in the equation) it is best to work directly with (53) and (54). To this end note the following simple results:

$$(uN)_x = uN_x + u_x N$$

$$(u_x N_x)_x = u_{xx} N_x + u_x N_{xx}$$

$$(uN)_{xx} = uN_{xx} + u_{xx} N + 2u_x N_x$$

$$uN_{xxx} + u_{xxx} N = (uN)_{xxx} - 3(u_x N_x)_x \tag{67}$$

$$6NN_x = 3(N^2)_x$$

$$2N_x N_{xx} = (N_x^2)_x$$

$$NN_{xxx} = (NN_{xx})_x - \frac{1}{2}(N_x^2)_x$$

Using these in (53) gives an alternative form for the integrable equation of motion in terms of the function $N(x,t)$:

$$\begin{aligned} &u_t + 6uu_x + u_{xxx} + \\ &\varepsilon[N_t + 6(Nu)_x + N_{xxx} - u_{xxt} - (u_x^2)_x - 2(uu_{xx})_x] + \\ &-\varepsilon^2[N_{xxt} + 2(uN)_{xxx} - 2(u_x N_x)_x - 3(N^2)_x] + \\ &-\varepsilon^3[(N_x^2)_x + 2(NN_{xx})_x] \end{aligned} \tag{68}$$

This equation is important because it is transparently integrable with respect to the spatial variable x. To find the travelling wave solution move to a frame of reference in which the wave is stationary and write:

$$u(x,t) = F(x'), \qquad x' = x - Ct \tag{69}$$

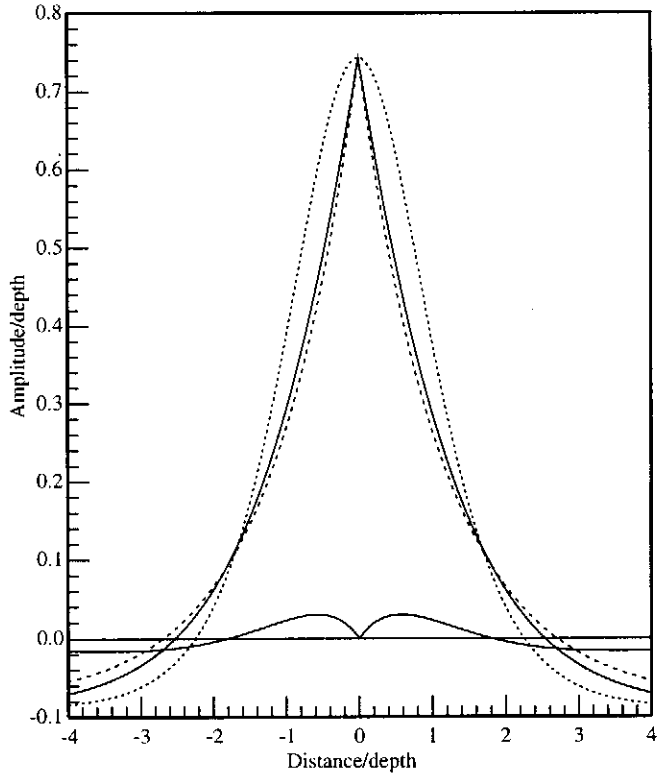

Fig. 12. Comparison of solitary waves for the highest wave (potential theory, solid peaked line), KdV soliton (dotted line), CH peakon (dashed line), exRW2 (graphically indistinguishable from the highest-wave curve). The difference between the highest wave and the CH solitary wave is shown as the double lobed curve near zero amplitude. The difference between the highest wave and the solitary wave for exRW2 is the horizontal line at zero amplitude (zero to 4 decimals).

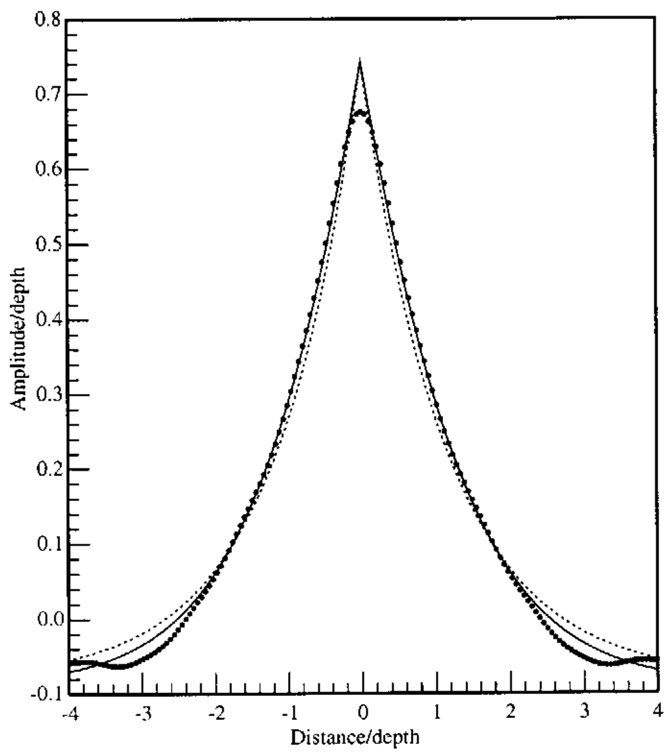

Fig. 13. Comparison of solitary wave for the CH equation (dotted line) and the exRW2 equation (solid line) with data (points).

so that

$$\frac{\partial}{\partial x}=\frac{\partial}{\partial x'}=(.)_{x'} \qquad \frac{\partial}{\partial t}=-C\frac{\partial}{\partial x'}=-C(.)_{x'}$$

where

$$N(u)=N[u(x,t)]=N[F(x')]= \\ =\lambda_1 F^2+\lambda_2 F_{x'x'}+\lambda_3 F_{x'}\partial^{-1}F \tag{70}$$

Use (69) in (68) (dropping the primes) and integrate once with respect to x to find:

$$\begin{aligned}&-CF+3F^2+F_{xx}+\\&+\varepsilon[6NF-CN+N_{xx}+CF_{xx}-F_x^2-2FF_{xx}]+\\&+\varepsilon^2[CN_{xx}-2(FN)_{xx}+2F_xN_x+3N^2]+\\&-\varepsilon^3[N_x^2+2NN_{xx}]+C_1=0\end{aligned} \tag{71}$$

where C_1 is an integration constant. Note that by setting $N=0$ we have the solitary wave for the Camassa-Holm equation; furthermore, by setting $\varepsilon=0$ we get the solitary wave for the KdV equation. Further analytical integration of (71) requires specification of the function $N(x,t)$. It is useful for numerical calculations to solve (71) for F_{xx}:

$$\begin{aligned}F_{xx}=\{&[A(u)F-3F^2+\varepsilon F_x^2+2\varepsilon^2N_xF_x]+\\&+\varepsilon[CN-N_{xx}]-\varepsilon^2[CN_{xx}+3N^2]+\\&+\varepsilon^3[N_x^2+2NN_{xx}]\}/D(u)\end{aligned} \tag{72}$$

where

$$A(u)=C-6\varepsilon N+2\varepsilon^2N_{xx}$$

$$D(u)=1+\varepsilon C-2\varepsilon F-2\varepsilon^2N$$

Numerically (72) is a second order equation which is to be integrated twice to obtain the solitary wave $F(x)$. Clearly the initial conditions $F(0)$, $F_x(0)$ need to be specified:

$$F(0)=a, \qquad F_x(0)=s$$

where a is the wave amplitude and s is the slope at the peak. A standard fifth-order Runge-Kutta routine has been employed for the numerical integration.

In Fig. 12 I show the solitary wave for the highest wave (solid line) as computed by the method of Evans and Ford (1994). Note that normalized coordinates are used, i.e. the wave amplitude and the spatial variable, x, are normalized by the depth. The highest wave computation is based upon a numerical solution of the full potential theory for water waves, e. g. equations (20) (see also Byatt-Smith, 1970; Longuet-Higgins, 1974; Byatt-Smith and Longuet-Higgins, 1976). This wave has of course the 120° Stokes angle at the crest. The solitary wave for the exRW2 equation is also graphed in Fig. 12; however, the results are equivalent to those for the highest wave to within graphical accuracy (four decimals). Thus, for practical purposes, with regard to the highest wave, the solitary waves of exRW2 compares well with the full water wave equations. Also shown in Fig. 12 (dashed line) is the peakon solution of Camassa and Holm (1993). This latter solution is found to have a peak angle somewhat less that the Stokes value (~109°) and to have tails which are somewhat higher than the highest wave for large x/h. For comparison I also give the soliton solution of the KdV equation (dotted line). These three theoretical solutions have much to say about the relative influence of nonlinear effects as one goes to higher order. It is surprising that the results for CH and exRW2 are so close to the highest solitary wave. One might not expect such good agreement in view of the large value of ε used in these calculations. This suggests that the lower order equations CH and exRW2 are perhaps more physically robust than one might have otherwise thought. Of course it comes as no surprise that the exRW2 equation is considerably more precise than the CH equation.

In Fig. 13 are comparisons of the theories with the wave data of Section 3. Shown are the theoretical curves given in Fig. 12 and the data (I have left out the soliton solution of the KdV equation as not being relevant for this highly nonlinear comparison). Note that the data points fall quite nicely on the curve for the highest wave (and for exRW2) except near the peak and for the small amplitudes in the tails. Note that no fitting process has been applied here; the theories and data have been simply graphed in normalized coordinates (amplitude/depth vs. x/depth) and a simple vertical shift has been made to improve agreement between theory and experiment. The difference between theory and data in the tails could easily be due to the presence of background radiation in the measured wave train. In this region there are physical effects which are not easily controllable in an experimental context. However, the discrepancy near the peak could well be due to the physical instability discussed earlier in Section 3. Water particles near the peak are susceptible to various kinds of external forcing not included in the water wave equations which could tend to erase the sharp peak predicted by potential theory. However one interprets the results given in Fig. 3, it seems plausible that one is addressing a physical situation which is rather more nonlinear than that described by the KdV equation. To within our ability to measure the shape of the "highest wave," the model equation exRW2 provides results which are consistent with our understanding of the water wave dynamics.

9 Extending the Order of Nonlinear Fourier Analysis Procedures

I now discuss extending the nonlinear Fourier analysis procedures discussed in Section 2 to include both the CH and the exRW2 equations. The approach is shown schematically in Fig. 14. The boxes labelled 2-5 contain the KdV nonlinear Fourier analysis procedures of Section 2. The spectral eigenvalue problem in box 2 has been modified, however, to include a factor $(1+4\varepsilon E)$ in front of the potential function $M(x)$. Thus by setting $\varepsilon=0$ and $M(x)=u(x,0)$ in box 2 reduces the spectral algorithm to that for the KdV equation. The preprocessor in box 1 shows this fact under the label "For KdV." In order to implement the CH equation one sets $\varepsilon=h^2/3$ and $M(x)=u(x,0)-\varepsilon u_{xx}(x,0)$. Likewise, to implement exRW2 one uses $M(x)=u-\varepsilon u_{xx}+\mu(N-\varepsilon N_{xx})$ for $N=\lambda_1 u^2+\lambda_2 u_{xx}+\lambda_3 u_x\partial^{-1}u$. Note that boxes 6, 7 are postprocessing steps which are necessary for recovering $u(x,t)$ from $M(x,t)$.

It is clear that extending the nonlinear Fourier analysis approach from the KdV equation to the CH equation and then to the exRW2 equation does not require much additional work in the computer programming. In particular the preprocessing step is quite simple as shown in box 1. Most of the work lies in the postprocessor steps 6, 7 where some knowledge of hyperelliptic and theta functions is required. I do not give here the details for these developments. However, it is important to point out that it is these results which lead to the fundamental superposition law discussed herein: Spectral solutions of exRW2 can be represented as a linear superposition of the travelling waves plus their mutual nonlinear interactions (Osborne, 1997).

Thus the approach given here for increasing the order of nonlinear Fourier analysis algorithms requires little additional mathematical and numerical effort compared to the fifteen-year long effort required to develop algorithms solely for the periodic KdV equation (Osborne, 1991; Osborne, 1995).

10 Summary and Discussion

I have suggested an approach for approximately integrating the second equation (W2) in the multiscale expansion of the Euler equations in the special case of irrotational, unidirectional surface wave dynamics in shallow water for 1 space and 1 time dimensions. A particular inverse scattering transform (IST) universal Lax pair is motivated on the basis of physical considerations and laboratory experiments; choice of

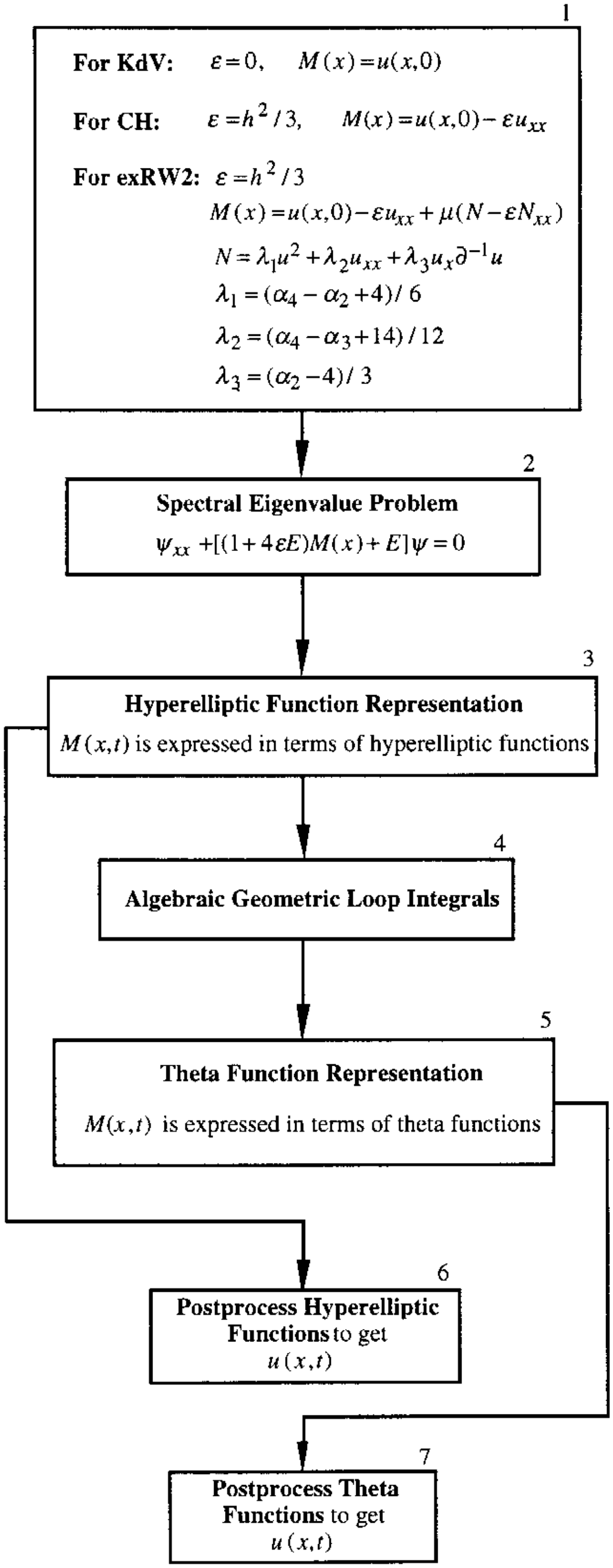

Fig. 14. Schematic of nonlinear Fourier analysis for the KdV, the CH and the exRW2 equations. Note that only one line of computer code (box 2) need be changed to render the KdV spectral algorithm capable of also computing spectra for the CH and exRW2 equations. One adds a preprocessor (box 1) to prepare the appropriate "potential," $M(x)$, for the spectral eigenvalue problem and two postprocessors (boxes 6, 7) for recovering $u(x,t)$ from the hyperelliptic and theta functions representations.

an arbitrary function in the formulation integrates a specific water wave equation. The particular equation W2 in the multiscale expansion is integrated formally to that order by IST, but, in order to insure integrability, the equation is appropriately regularized and includes precisely calculated additional terms at even higher order. The universal Lax pair allows computation of these results by application of the compatibility condition.

It should be pointed out that there are no guarantees that the integrable equation exRW2 is as physically useful, for example, as the KdV equation. In order to assess whether exRW2 has physical merit I have studied its linear dispersion relation and solitary wave solution and compared these results to those for the water wave equations. The comparison is sufficiently favorable so as to suggest that future research should focus on this new equation or on related equations. For example, study of the full spectral structure of exRW2 will reveal whether it is in some sense close to the full unidirectional water wave dynamics (Osborne, 1997).

Here is a list of observations of the new approach given herein for approximately integrating nonlinear wave equations to higher order:

(1) The universal wave equation (48) is completely integrable by the inverse scattering transform universal Lax pair (51) and hence exRW2 (59) (constructed using (48), (54)) is also integrable by IST.

(2) The expanded form of the universal wave equation (53) contains both the KdV and the CH equations. This is seen by letting $N \to 0$ to get the CH equation and by subsequently letting $\varepsilon \to 0$ to get the KdV equation. The coefficients λ_i have been adjusted in (54) to give precisely the α_i in (3b) after the wave equation (59) is deregularized and truncated to $O(\varepsilon)$ (see (57), (58)).

(3) The fact that one uses the CH equation as a spring board to higher order integrability via the Lax pair (51) suggests that the CH equation is itself a generic wave equation from the point of view of the mathematical physics. As seen in the example above, adding an arbitrary function to solutions of the KdV equation (while maintaining integrability) is not simple, i.e. one is forced to use $-\varepsilon u_{xx}$ or more generally (26). For the CH equation, however, one is able to add an *arbitrary function* $N(x,t)$ which easily leads to a higher order integrable wave equation (48). It may therefore be worthwhile in future efforts to seek out universal Lax pairs of the form (51) for other physical applications such as the NLS equation and for searches for higher order integrability in higher dimensions.

(4) While it is possible to integrate exRW2 using the Lax pair (51) it should be noted that the resultant *regularized wave equation* (59) contains *specific higher order terms at* $O(\varepsilon^2)$ *and* $O(\varepsilon^3)$ which insure the integrability. Thus, using the method given here one is able to integrate (3b) only by adding particular terms at $O(\varepsilon^2)$ and $O(\varepsilon^3)$ and by retaining the appropriate regularized structure given in (59) (see Fig. 15 and discussion below).

(5) The results presented here are completely consistent with Kodama (1985a,b) and Fokas and Liu (1996) in which they suggest that higher order equations should be asymptotically integrable. Particular exploitation of these ideas is pursued here by *formally integrating* nearby nonlinear wave equations. I develop an approach for forcing integrability via generalization of the relevant Lax pair. I have integrated an equation *equivalent* to (3b), namely (59). Furthermore, the approach appears formally extendible to yet higher order by an appropriate selection of the function N in the universal equation (48).

The advantage of addressing the integrable wave equation (48), is that we have the complete spectral structure (51) to give us understanding of the underlying physical behavior. The fact that the actual equation under study, (3b), is not very far away from the integrable behavior of (59), for suitably chosen ε, indicates that we have approximately integrated the higher order wave motion. In this way we are able to study (3b) by addressing the exactly integrable structure of (59). These issues are discussed in detail in Osborne et al. (1997) where the appropriateness of (59) as an approximate descriptor of the water wave equations is addressed in full detail.

A schematic of the Whitham hierarchy, Wn, and the integrable hierarchy, exRWn, is given in Fig. 15. In columnar fashion, along the left hand side, are the members of the multiscale expansion W0, W1, W2 and W3. On the right hand side are the members of the hierarchy addressed herein, namely, the extended, appropriately regularized multiscale expansion which interleaves the sequence Wn, although the existence of integrable equation exRW3 has yet to be established.

How can a wave equation which is as complicated as (59), even though it is integrable, be useful in the study of the nonlinear dynamics of water waves? Are not the requisite necessary mathematical analyses too complicated and lacking in transparency to be useful for physical understanding? To fully address this question it is worth pointing out that many of the properties of the integrable wave equation (59) which are of interest here are determined from the associated spectral problem (51), which can be summarized as

$$\psi_{xx} + [(1+4\varepsilon E)M(x) + E]\psi = 0 \tag{73}$$

where

$$M(x) = u - \varepsilon u_{xx} + \mu(N - \varepsilon N_{xx}) \tag{74a}$$

$$N = \lambda_1 u^2 + \lambda_2 u_{xx} + \lambda_3 u_x \partial^{-1} u \tag{74b}$$

with $\lambda_1 = (\alpha_4 - \alpha_2 + 4)/6$, $\lambda_2 = (\alpha_4 - \alpha_3 + 14)/12$ and $\lambda_3 = (\alpha_2 - 4)/3$. Thus (73) and (74) (together with the second equation of (51)) integrate equation (59). By selecting the appropriate values for α_i one can, for example, approximately integrate the surface wave equation W2 or the related equation for internal wave motions (Lee and Beardsley, 1974).

As noted above, by setting $\mu = 0$ in (73), (74), one gets the spectral problem for the CH equation (44); by further setting $\varepsilon = 0$ we have the KdV spectral problem (36). Future studies of the Lax pair (51) will likely reveal much more about the integrable properties of exRW2 (59). The simplicity of these results means that the requisite implementation of (73), (74) as a tool for spectral analysis is straightforward, in spite of the complexity of (59).

Note, further, that I have interpreted the above results (73), (74) as arising due to the simple transformation $u \to u + \mu N(u)$ in the CH Lax pair (44). However, an alternative interpretation is that one makes the transformation $u \to u - \varepsilon u_{xx} + \mu(N - \varepsilon N_{xx})$ in the KdV Lax pair ((36), (37), where one also has to introduce the spectral term $(1 + 4\varepsilon E)$ which multiplies the potential $M(x)$ in (51)). From my own point of view the use of the CH equation as a route to higher order integrability is preferred due to the transparent simplicity of the derivation given in Section 6.4; this perspective provides the spectral representation (73), (74) via the route given by proposition 2 and (28). One may alternatively discard this point of view and interpret (73), (74) as a single leap to higher order from the KdV equation itself; this perspective provides the spectral representation (73) via the route given by Proposition 1 and (26).

Finally I would like to briefly summarize the method used here to approximately integrate the unidirectional water wave equations to higher order. The approach proceeds as follows. First begin with a particular *seed equation*, i.e. the leading order nonlinear wave equation integrable by the inverse scattering transform, namely the KdV equation. Consider the Lax pair for this equation and make the transformation: $u \to u + \varepsilon P(u)$; ε is some arbitrary parameter whose physical value is to be established later on the basis of the multiscale expansion. The function $P(u)$ is established during application of the compatibility conditions in order to insure integrability. By generalizing the Lax pair for the seed equation in a simple way one succeeds in deriving and integrating the Camassa-Holm equation. This equation, which is of higher order than the seed equation, is deficient in certain terms necessary for matching to the higher order equation W2 obtained from the multiscale expansion of the water wave equations. A property of the Camassa-Holm equation is that one is able to add *any* arbitrary function to its solution and to

W0 [linearized KdV equation]

W1 [KdV equation]

$\longrightarrow$ Camassa-Holm = KdV + $O(\varepsilon)$

W2 [KdV + $O(\varepsilon)$]

$\longrightarrow$ exRW2 = Reg(W2) + $O(\varepsilon^2, \varepsilon^3)$

W3 [KdV + $O(\varepsilon)$+ $O(\varepsilon^2)$]

$\longrightarrow$ exRW3 = Reg(W3) + $O(\varepsilon^3, \varepsilon^4, \varepsilon^5)$

Fig. 15. Schematic of the relationship between the Whitham hierarchy for the multiscale expansion of the irrotational, unidirectional water wave equations (Wn) and the associated extended, regularized hierarchy (exRWn) discussed herein.

thereby obtain an integrable wave equation. I call this approach the *method of universal Lax pairs*. The latter relations can then be used to *target* on a particular higher order equation from the multiscale expansion. In this way one obtains an integrable wave equation which *matches* at a particular order an equation in the multiscale expansion. This integrable equation is regularized and has higher order terms, beyond the order of the target equation, which insure integrability. One then checks the physical validity of the new integrable equation. Some important tests are: (1) compare the linear dispersion relation to that for linearized water waves, (2) compare the solitary wave solution to that for the full water wave equations, (3) compare to laboratory data and (4) compare the behavior of the inverse scattering transform spectrum using results from a numerical code for the full water wave equations (this work is in progress).

The method presented here appears *formally* applicable for integrating the equations in the multiscale expansion for water waves to arbitrary order. However, it is presently unknown as to whether the approach will fail at some particular order, say due to the possible presence of inelastic effects, chaos or other unforeseen nonlinear phenomenology. Should the method eventually be found to extend to arbitrarily high order, then it goes without saying that, in some sense, complete integration of the appropriately extended and regularized Whitham multiscale expansion of the unidirectional, water wave equations in the *long wave regime* (i.e. for wave numbers smaller than the Benjamin-Fier instability, $kh = 1.36$) would be virtually assured.

Acknowledgements. This work has been funded in part by the Italian National Research Council (CNR) and by the Italian Council of University Funds (CUN). We are also grateful to the Consortium for the Development of the Offshore (CDO) for providing support for software development of nonlinear Fourier analysis methods and for laboratory and Adriatic Sea surface wave experiments. I thank W. A. B. Evans for aid and advice in computing the highest wave. I also thank Roberto Camassa for patiently clarifying my thoughts with regard to a number of aspects of the present work.

References

Ablowitz, M. J. and Segur, H., *Solitons and the Inverse Scattering Transform*, SIAM, Philadelphia, 1981.

Abramowitz, M. and Stegun, I. A., *Handbook of Mathematical Functions*, National Bureau of Standards Applied Mathematics Series 55, 1964.

Benjamin, T. B., Bona, J. L. and Mahoney, J. J., Model equations for long waves in nonlinear dispersive systems, *Philos. Trans. R. Soc. London Ser. A 272*, 47, 1972.

Boyd, J. P., The double cnoidal wave of the Korteweg-deVries equation: An overview, *J. Math. Phys. 25*, 3402, 1984.

Boyd, J. P., Perturbation series for the double cnoidal wave of the Korteweg-deVries equation, *J. Math. Phys. 25*, 3402, 1984.

Boyd, J. P., The special modular transformation for polycnoidal waves of the Korteweg-deVries equation, *J. Math. Phys. 25*, 3415, 1984.

Byatt-Smith, J. G. B., An exact integral equation for steady surface waves, *Proc. Roy. Soc. Lond., A315*, 405-418, 1970.

Byatt-Smith, J. G. B. and Longuet-Higgins, M. S., On the speed and profile of steep solitary waves, *Proc. Roy. Soc. Lond., A350*, 175-189, 1976.

Calogero, F. and Degasperis, A., *Spectral Transform and Solitons*, North-Holland, Amsterdam, 1982.

Camassa, R. and Holm, D. D., A completely integrable dispersive shallow water equation with peaked solitons, *Phys. Rev. Lett. 71*, 1661-1664, 1993.

Camassa, R., Holm, D. D. and Hyman, J. M., A new integrable shallow water equation, *Advances in Applied Mechanics 31*, 1-33, 1994.

Date, E. and Tanaka, S., Periodic multisoliton solutions of Korteweg-deVries equation and Toda lattice, *Prog. Theor. Phys. Supp. 59*, 107, 1976.

Dodd, R. K., Eilbeck, J. C., Gibbon, J. D. and Morris, H. C., *Solitons and Nonlinear Wave Equations*, Academic Press, London, 1982.

Dubrovin, B. A., Matveev, V. B. and Novikov, S. P., Nonlinear equations of Korteweg-deVries type, finite zoned linear operators and Abelian varieties, *Russ. Math. Surv. 31*, 59, 1976.

Drazin, P. G. and Johnson, R. S., *Solitons: an introduction*, Cambridge University Press, Cambridge and New York, 1989.

Evans, W. A. B. and Ford, M. J., An exact integral equation for solitary waves, preprint, 1994.

Ferguson, W. E., Jr., Flaschka, H. and McLaughlin, D. W., Nonlinear normal modes for the Toda chain, *J. Comput. Phys. 45*, 157, 1982.

Fokas, A. S. and Liu, Q. M., Asymptotic integrability of water waves, *Phys. Rev. Lett. 77*, 2347-2351, 1996.

Fordy, A. P. (Ed.), *Soliton theory: a survey of results*, Manchester University Press, Manchester and New York, 1990.

Flaschka, H. and McLaughlin, D. W., Cononically conjugate variables for KdV and Toda lattice under periodic boundary conditions, *Prog. Theor. Phys. 55*, 438, 1976.

Fuchssteiner, B., On integrable systems with higher order corrections, *Prog. Theor. Phys. 65*, 861-876, 1981.

Fuchssteiner, B. and Fokas, A. S., Symplectic structures, their Backlund transformations and hereditary symmetries, *Physica (Amsterdam) 4D*, 47-66, 1981.

Gardner, C. S., Greene, J. M., Kruskal, M. D. and Miura, R. M., Method for solving the Korteweg-deVries equation, *Phys. Rev. Lett. 19*, 1095-1098, 1967.

Its, A. R. and Matveev, V. B., Hill's operator with finitely many gaps, *Funct. Anal. Appl. 9*, 65, 1975.

Kaup, D. J., A higher-order water-wave equation and the method for solving it, *Prog. Theor. Phys., 54*, 396-408.

Kaup, D. J., private communication, 1986.

Kinsman, B., *Wind Waves*, Prentice-Hall, Englewood Cliffs, New Jersey, 1965.

Kodama, Y., Normal forms for weakly dispersive wave equations, *Phys. Lett. 112A(5)*, 193-196, 1985; On integrable systems with higher order corrections, *Phys. Lett. 112A(6)*, 245-249, 1985.

Korteweg, D. J. and deVries, G., On the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves, *Philos. Mag. Ser. Vol. 5(39)*, 422-443, 1895.

Lax, P. D., Integrals of nonlinear equations of evolution and solitary waves, *Comm. Pure Appl. Math. 21*, 467-490, 1968.

Lee, C.-Y. and Beardsley, R. C., The generation of long nonlinear internal waves in a weakly stratified shear flow, *J. Geophys. Res. 79(3a)*, 453-462, 1974.

Longuet-Higgins, M. S., On the mass, momentum, energy and circulation of a solitary wave, *Proc. Roy. Soc. Lond., A337*, 1-13, 1974.

Longuet-Higgins, M. S. and Fenton, J. D., On the mass, momentum, energy and circulation of a solitary wave. II., *Proc. Roy. Soc. Lond., A340*, 471-493, 1974.

McKean, H. P. and Trubowitz, E., Hill's operator and hyperelliptic function theory in the presence of infinitely many branch points, *Comm. Pure Appl. Math. 29*, 143, 1976.

Miles, J. W., Solitary waves, *Ann. Rev. Fluid Mech. 12*, 11-43, 1980.

Munk, W. H., Solitary waves, *N. Y. Acad. Sci. 51*, 376, 1949.

Newell, A. C., *Solitons in Mathematics and Physics*, SIAM, Philadelphia, 1985.

Novikov, S., Manakov, S. V., Pitaevskii, L. P., Zakharov, V. E., *Theory of Solitons-The Inverse Scattering Method*, Consultants Bureau, New York and London, 1984.

Osborne, A. R. and Burch, T. L., Internal solitons in the Andaman Sea, *Science 208*, 451-460, 1980.

Osborne, A. R., Nonlinear Fourier analysis, in *Nonlinear Topics in Ocean Physics*, edited by A. R. Osborne, North-Holland, Amsterdam, 1991.

Osborne, A. R., Numerical construction of nonlinear wave-train solutions of the periodic Korteweg-deVries equation, *Phys. Rev. E 48(1)*, 296-309, 1993.

Osborne, A. R., Solitons in the periodic Korteweg-deVries equation, the θ-function representation, and the analysis of nonlinear, stochastic wave trains, *Phys. Rev. E 52(1)*, 1105-1122, 1995.

Osborne, A. R., 1997, Integrating higher order wave equations in multiscale expansions of the water wave equations, in preparation.

Osborne, A. R. and Petti, M., Laboratory-generated, shallow-water surface waves: Analysis using the periodic, inverse scattering transform, *Phys. Fluids, Vol. 6(17)*, 1727-1744, 1994.

Osborne, A. R., Petti, M., Serio, M. and Bergamasco, L., The nonlinear dynamics of a sine wave propagating above a linearly decreasing bottom profile, to be published, 1997.

Osborne, A. R., Solutions to extended order, nonlinear wave equations, in preparation, 1997.

Sarpkaya, T. and Isaacson, M., *Mechanics of Wave Forces on Offshore Structures*, Van Nostrand Reinhold, New York, 1981.

Svendsen, Ib. A. and Hansen, J. B., On the deformation of periodic long waves over a gently sloping bottom, *J. Fluid Mech., 87(3)*, 433-448, 1978.

Tanaka, M., The stability of solitary waves, *Phys. Fluids, 29(3)*, 650-655, 1986.

Weigel, R., *Oceanographical Engineering*, Prentice-Hall, Englewood Cliffs, N. J., 1964.

Whitham, G. B., *Linear and Nonlinear Waves*, John Wiley, New York, 1974.